Test 1 Review College Algebra Mike Huff

Numbers

Sets of Numbers: Natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.

Percent change: $\frac{c_2 - c_1}{c_1} \times 100\%$

Exercise 1.1: Find the percent change in a quantity if it changes from $c_1 = 25$ to $c_2 = 40$.

Answer: $\frac{40-25}{25} \times 100\% = 60\%$

Scientific Notation: $a \times 10^n$ where a is an integer such that $1 \le |a| < 10$.

Exercise 1.2:

a) The distance to sun is 93,000,000 miles. Express this in scientific notation.
   Answer: 9.3×10^7 miles
b) One lightyear is $9.46 \times 10^{12}\, km$. Express this in standard notation.
   Answer: $9,460,000,000,000\, km$

Statistics

One Variable Data

Measures of Center

Definition: The **mean** (average) of a set of numbers is the sum of the numbers divided by the count of the numbers. $\overline{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$

Definition: The median of a set of numbers is the middle score in an *ordered* list.

Measures of Spread

Definitions: The **minimum** of a set of numbers is the smallest value in a list of the data. The **maximum** of a set of numbers is the largest value in a list of the data. The **range** of a set of numbers is the difference between the maximum score and the minimum score:

range = maximum score - minimum score

Exercise 1.3:

a) Find the mean of the set of numbers $\{91, 78, 64, 88, 92\}$. Answer: $\overline{x} = 82.6$
b) Find the median of the set of numbers $\{91, 78, 64, 88, 92\}$. Answer: $M = 88$
c) Find the range of the set of numbers $\{91, 78, 64, 88, 92\}$.Answer: $92 - 64 = 28$

All of these can be found using the one variable stats command on the TI-83

To find one variable statistics: enter the following:
1. Put the data in List 1 (L1) using the STAT EDIT menu

⊆ gives you the lists. If there is data in a list scroll up using the } button until the name of the list is highlighted. Push ⊆, the list should now be empty. Enter the data in the list by typing in one number at a time and pushing ⊆ after each one. Once the data is entered, return to the main screen using Quit (ψζ)

2. Find the one variable statistics: ~ 1 ⊆

$\bar{x}$ is the mean
n is the number of scores - Use to see the rest of the results
minX is the minimum score
maxX is the maximum score
Med is the median

Two Variable Data

Definition: A **relation** is any collection of ordered pairs.

Example:

Year	1994	1995	1996	1997	1998
CDs (% share)	58.4	65.0	68.4	70.2	74.8

This table defines a relation

Definition: The **domain** of a relation is the set of all possible values of the first coordinate (independent variable).

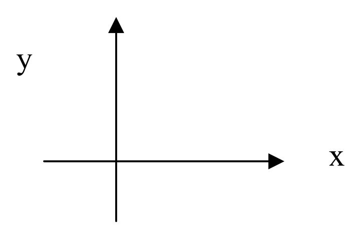
Definition: The **range** of a relation is the set of all possible values for the second coordinate (dependent variable).

Example: For the relation above, the domain is $\{1994, 1995, 1996, 1997, 1998\}$ and the range is $\{58.4, 65.0, 68.4, 70.2, 74.8\}$.

Exercise 1.4: Find the domain and range of the relation $f = \{(0,1),(1,2),(2,3)\}$

Answer: The domain is $\{0,1,2\}$ and the range is $\{1,2,3\}$.

To visualize data with two variables we use the **Cartesian plane**. This consists of two perpendicular axes (x-axis and the y-axis). The axes divide the plane into four **quadrants**.



The Distance Formula and Related Ideas

<u>The Pythagorean Theorem</u>: In a right triangle, the sum of the squares of the sides equals the square of the hypotenuse: $a^2 + b^2 = c^2$.

<u>Distance Between Points</u>: Given any two points (x_1, y_1) and (x_2, y_2), the distance (d) between the two points can be found by using the Pythagorean Theorem. The result is called the Distance Formula: **Distance Formula:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

<u>The Midpoint Formula:</u> Given any two points (x_1, y_1) and (x_2, y_2), the midpoint of the line segment joining the two points is the point M whose coordinates are given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Exercise 1.5: Find the distance between the following pairs of points.

a) $(-1,5)$ and $(3,7)$ Answer: $d = \sqrt{(3-(-1))^2 + (7-5)^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

b) $(3,-5)$ and $(-2,-1)$ Answer:

$$d = \sqrt{(-2-3)^2 + (-1-(-5))^2} = \sqrt{(-5)^2 + (4)^2} = \sqrt{41}$$

c) Find the midpoint of the line segment through $(-2,4)$ and $(1,5)$. **Answer:**

$$\left(-\frac{1}{2}, \frac{9}{2}\right)$$

<u>Circles</u>: A circle is a set of points a fixed distance (the radius) from a given point. The equation of a circle with radius r centered at the origin is given by $x^2 + y^2 = r^2$. If the circle is centered at (h,k), then the equation is $(x-h)^2 + (y-k)^2 = r^2$

Exercise 1.6: Put the equation of the circle in standard form and graph.

a) $x^2 + 6x + y^2 - 4y = 0$ Answer: $(x+3)^2 + (y-2)^2 = 13$, center $(-3,2)$, $r = \sqrt{13}$

b) $x^2 - 8x + 3 + y^2 + 6y - 3 = 0$

Answer: $(x-4)^2 + (y+3)^2 = 25$, center $(4,-3)$, $r = 5$

Functions

Definition: A **function** is a collection of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinates. We can also say that a function is a correspondence between two sets such that each element in the first set is paired with exactly one element in the second set.

Definition: The **domain** of a function is the set of all possible values of the first coordinate (independent variable).

Definition: The **range** of a function is the set of all possible values for the second coordinate (dependent variable). If *x* is the input value and the function *f* is given, then **the value of *f* at *x*** (or *f* of *x*) is $f(x)$. Note: A function is really a set: $f = \{(x, y) | y = f(x)\}$.

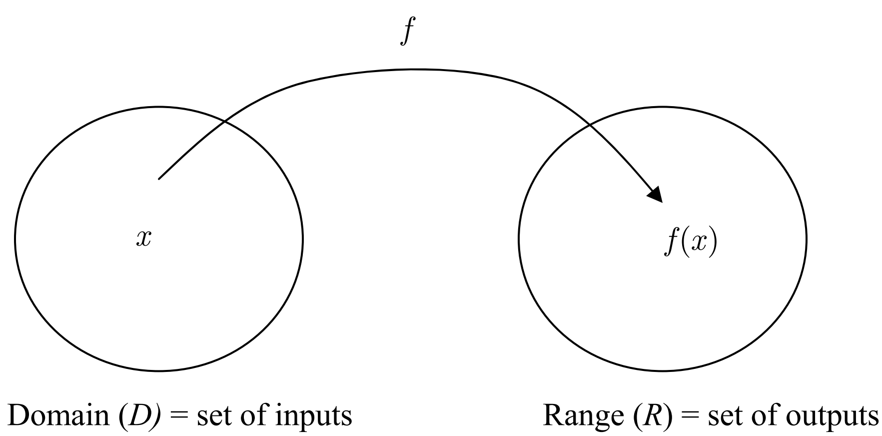
Example:

a) $f = \{(0,1),(1,2),(2,3)\}$ is a function
b) $g = \{(0,1),(0,2),(2,3)\}$ is a relation but is not a function since the number 0 gets paired with two different output values.
c) The domain of the function *f* defined in a) is $\{0,1,2\}$

Exercise 1.7: Determine if the following relations are functions.

a) $f = \{(0,1),(1,2),(2,3),(3,2),(1,3)\}$ Answer: No, $y = 2$ or 3 when $x = 1$
b) $g = \{(0,1),(1,2),(2,3),(3,4),(4,5)\}$ Answer: yes

Definition: The **graph** of a function *f* with domain *D* is the set $\{(x, f(x)) | x \text{ is an element of } D\}$. That is, the graph of a function is a set of ordered pairs.



In this course we will often make use of the four representations of functions.

The four representations of functions: $\begin{cases} \textit{verbal} \\ \textit{graphic} \\ \textit{numeric} \\ \textit{symbolic} \end{cases}$

Theorem: Vertical Line Test: If a vertical line cannot be drawn so that it touches a graph in more than one point, then the graph is the graph of a function.

Function Notation

$f(x)$ denotes the value of the function *f* at *x*. It is a number in the range of the function *f*. If $y = f(x)$ then we say that *y* is *f* of *x*. To evaluate a function, substitute the appropriate value of *x* into the formula.

Example: Let $f(x) = 2x + 1$ and $g(x) = -3x^2 + x$. Find $f(-2)$ and $g(-2)$ symbolically.

a) $f(-2) = 2(-2) + 1 = -4 + 1 = -3$
b) $g(-2) = -3(-2)^2 + (-2) = -3 \cdot 4 - 2 = -14$

Exercise 1.8: Let $f(x) = -5x + 3$ and $g(x) = -2x^2 - 3x$. Find the following symbolically:

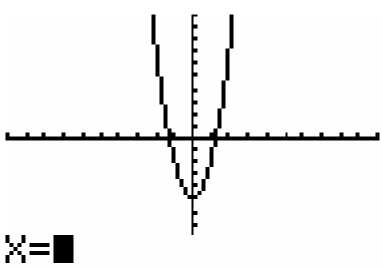
a) $f(-1)$ Answer: $f(-1) = 8$
b) $f(4)$ Answer: $f(4) = -17$
c) $g(-3)$ Answer: $g(-3) = -9$
d) $g(2)$ Answer: $g(2) = -14$

It is possible to evaluate a function using the value command on the TI-83 calc menu.

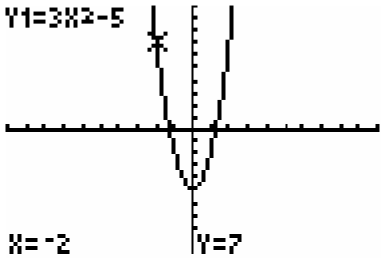
Example: Find $f(-2)$ if $f(x) = 3x^2 - 5$ using the TI-83.

1. Enter the function using the o menu.
2. Graph the function in a suitable window. The window must contain the point at which you are trying to evaluate the function.
3. From the Calc menu(ψρ) choose 1: Value

   Using the standard viewing window [-10,10,1] by [-10,10,1] , you should see something like this:



   The prompt is asking you for the *x* value. Type in -2 and hit ⊆. You should get:



This shows us that $f(-2) = 7$ and the point is also marked with an X.

If no domain is listed for a particular function, then the domain is the set of real numbers for which its symbolic representation is defined. This is called the **implied domain**.

Exercise 1.9: Find the implied domains of the following functions.

a) $g(x) = \dfrac{x+2}{x-5}$
Answer: $\{x | x \neq 5\}$

b) $f(x) = \sqrt{x+1}$
Answer: $\{x | x \geq -1\}$

c) $h(x) = \dfrac{1}{\sqrt{x-1}}$
Answer: $\{x | x > 1\}$

d) $F(x) = \dfrac{x+2}{\sqrt{2-x}}$
Answer: $\{x | x < 2\}$

Functions Specified by Equations

In an equation in two variables, if to each value of the independent (input) variable there corresponds exactly one value of the dependent (output) variable, then the equation defines a function.

Examples: Equations that define functions.

a) $y = 3x + 5$
b) $y = \sqrt{x+1}$
c) $y = x^2$
d) $y = x^3$

If there is any value of the independent variable to which there corresponds more than one value of the dependent variable, then the equation does not define a function.

Examples: Equations that *do not* define functions.

a) $|y| = 3x + 5$ For example, if $x = -1$, then we have $|y| = 2$. This has two solutions $y = \pm 2$. Since a single value of *x* gives us two values for *y,* the equation does not define a function.

b) $y^2 = x$ For example, if $x = 4$, then we have $y^2 = 4$. This has two solutions $y = \pm 2$. Since a single value of *x* gives us two values for *y,* the equation does not define a function.

Exercise 1.10: Determine if the following equations determine functions.

a) $|y| = x$ Answer: no

b) $y^2 = x$ Answer: no

c) $y^2 - x^2 = 9$ Answer: no

Graphing Functions by Hand

Exercise 1.11: Graph the following functions by plotting points. First, determine a numerical representation and then use that to give a graphical representation. Make a table using the x-values: $x = -2, -1, 0, 1, 2$.

a) $f(x) = |2x|$

b) $f(x) = \sqrt{2x + 1}$

c) $f(x) = x^2 - 4$

d) Graph all of the above functions using a graphing calculator.

Types of Functions and Their Rates of Change

Constant Functions

Suppose you recently got a new job. At your new job your salary is $\$2,000$ a week regardless of how many hours you work. You could create a table that shows your salary as a function of the number of hours you work in a week:

Hours worked	40	42	44	46	48	50	52
Dollars earned	2,000	2,000	2,000	2,000	2,000	2,000	2,000

Since the output value never changes, this is called a *constant function*.

Definition: A function of the form $f(x) = b$ is called a **constant function**.

If the domain of a function f consists of only a finite number of values, we call the function a **discrete function**. If the graph of a function can be drawn without lifting our pencil from the paper, then the function is said to be a **continuous function**.

Linear Functions

Suppose a car is traveling from a location 10 miles from Austin to New Orleans at 65 miles per hour. The distance between Austin and the car are listed in the table below for various times.

Time (hours)	0	1	2	3	4	5	6
Distance (miles)	10	75	140	205	270	335	400

This data is modeled by the set of ordered pairs

$$f = \{(0,10),(1,75),(2,140),(3,205),(4,270),(5,335),(6,400)\}$$

This set is a function but it is not a constant function.

Exercise 1.12: Make a scatterplot of the data in the table above.

Definition: A function of the form $f(x) = ax + b$ is called a **linear function** where a, b are real numbers with $a \neq 0$. If $a = 0$ then we have a constant function.

Hours worked	40	42	44	46	48	50	52
Dollars earned	2,000	2,000	2,000	2,000	2,000	2,000	2,000

Slope as a Rate of Change

Definition: The **slope** of a line through two points is the ratio of the vertical change to the horizontal change. The slope of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\text{slope} = m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Exercise 1.13: Find the slope of the line passing through the points $(-3,2)$ and $(2,4)$.

Answer: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-2}{2-(-3)} = \dfrac{2}{5}$

Exercise 1.14: The graphing calculator can be used to quickly graph most functions. However, we have to be careful about the appearance of the graph in the window. Graph the function $f(x) = x$ in the standard viewing rectangle. Does the change in y appear to be the same as the change in x? Now use a square viewing rectangle to graph the function $f(x) = x$. Which window gives us a more realistic looking graph? Hint: Use ZOOM square.

Nonlinear Functions

Definition: If a function is not linear, then it is said to be a **nonlinear function.** The graph of a nonlinear function is not the graph of a straight line.

Example: Recognizing linear and nonlinear data and functions

Does the following table represent a linear or nonlinear function?

x	-2	-1	0	1	2
y	1.2	2.3	3.4	4.5	5.6

The function is linear if the slope is the same between each pair of points.

For the first pair of points, we have $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2.3 - 1.2}{-1-(-2)} = \dfrac{1.1}{-1+2} = 1.1$. A similar calculation shows that the slope (rate of change) remains constant between each pair of points, so the function is linear.

Exercise 1.15: **Recognizing linear and nonlinear data and functions**

a) Is the following data linear or nonlinear?

x	-4	0	1	2	5
y	5.0	3.0	2.5	2.0	0.5

b) Is the following data linear or nonlinear?

x	-4	-2	0	2	4
y	1.0	-0.5	-2.0	-3.5	-5.0

Answer: a) linear because average rate of change is a constant –0.5
b) linear because average rate of change is a constant –0.75

Average Rate of Change

If a function is nonlinear, then we do not have a single slope. A line that crosses the graph of a function at two points (x_1, y_1) and (x_2, y_2) is called a **secant line.** The slope of the secant line is called the average rate of change of f from x_1 to x_2.

Definition: If (x_1, y_1) and (x_2, y_2) are two distinct points on the graph of a function f, then the **average rate of change of f from x_1 to x_2** is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

That means that the average rate of change equals the slope of the line passing through the two points.

Exercise 1.16: Average Rate of Change

a) Find the average rate of change of $f(x) = 5x - 3$ from $x_1 = 2$ to $x_2 = 5$.

b) Find the average rate of change of $f(x) = 0.5x^2 - 3$ from $x_1 = -1.2$ to $x_2 = 3.4$.

Answer: a) $ARC = 5$

b) $ARC = \dfrac{\left(0.5(3.4)^2 - 3\right) - \left(0.5(-1.2)^2 - 3\right)}{3.4 - (-1.2)} = \dfrac{2.78 - (-2.28)}{4.6} = \dfrac{5.06}{4.6} = 1.1$

Graphing Lines

Exercise 1.17: **Graphing Lines**

a) $2x + 5y = 10$ Graph using intercepts.

b) $4x - 3y = 12$ Graph using point plotting.

<u>Special Cases</u>:

1. The graph of $x = a$ is a **vertical line** that passes through $(a, 0)$.
2. The graph of $y = b$ is a **horizontal line** that passes through $(0, b)$.

Definition: The **slope** of a line through two points is the ratio of the vertical change to the horizontal change. The slope of the line through the points (x_1, y_1) and (x_2, y_2) is

slope = $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Exercise 1.18: **Finding the slope of the line between two points**

a) Find the slope of the line between the points $(1,2)$ and $(3,6)$ **answer:** $m = 2$
b) Find the slope of the line containing the points (-3,4) and (2,6). **answer:** $m = 2/5$

Forms of the Equation of a Line

Definition: The **slope-intercept form** of the equation of a line is $y = mx + b$.

Exercise 1.19: **Slope-intercept equation of lines**

a) Give an equation for the line that has slope $-\frac{1}{2}$ and a y-intercept of –5.
b) Determine the slope and the y-intercept of the following: $5x - 10y = 25$

Answers: a) $y = -\frac{1}{2}x - 5$ b) $m = 1/2$, y-intercept is $(0, 5/2)$

Definition: The **point-slope form** of the equation of a line is $y - y_1 = m(x - x_1)$.

This equation comes from the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. We now assume we only know one point (x_1, y_1). The slope formula then becomes $m = \frac{y - y_1}{x - x_1}$. This can be rearranged to get the point-slope form: $y - y_1 = m(x - x_1)$.

Exercise 1.20: **Point-slope equation of lines**

a) Give an equation for the line between the points $(1,2)$ and $(3,6)$.
b) A company manufacturing surfboards has fixed costs of \$450 per day and total costs of \$9,450 per day at a daily output of 25 boards. Write an equation that relates cost to the number of boards made.
Answers: a) $y = 2(x - 1) + 2$ b) $y = 360x + 450$

Definitions: Two lines are **parallel lines** if they have the same slope and different y-intercepts. Two lines are **perpendicular** if they have slopes whose product is -1.

Suppose two lines are given with slopes m_1 and m_2 respectively. Then the lines are parallel if $m_1 = m_2$ and perpendicular if $m_1 \cdot m_2 = -1$. The slopes of perpendicular lines are negative reciprocals since $m_1 = \frac{-1}{m_2}$.

Exercise 1.21:

a) Give an equation for the line through the point $(1,2)$ parallel to the line $2x - y = 7$.
b) Give an equation for the line through the point $(-1,3)$ perpendicular to the line $2x + y = 7$.
c) Give an equation for the line through the point $(-3,-2)$ parallel to the line $5x - 3y = 11$.
d) Give an equation for the line through the point $(3,-4)$ perpendicular to the line $2x - 3y = 5$.

Answers:

a) $y = 2(x-1)+2$

b) $y = \frac{1}{2}(x+1)+3$

c) $y = \frac{5}{3}(x+3)-2$

d) $y = \frac{-3}{2}(x-3)-4$

Direct Variation

We say that y varies directly as x if $y = kx$. The letter k is called the constant of proportionality.

Example: Suppose a teacher can grade 25 test papers in three hours. If we let x be the time grading and y equal the number of papers graded, then we can write $25 = k \cdot 3$. If we solve for k, we get $k = \frac{25}{3}$. The equation of variation is then $y = \frac{25}{3}x$

How many tests can she grade in 9 hours? Let $x = 9$, then we have

$y = \frac{25}{3}(9) = 75$. So she can grade 75 papers in 9 hours.

Solving Linear Equations in One Variable

Exercise 1.22

a) $6x + 3 = 3x - 11$

b) $f(x) = 3(x-3) + 2(x-6)$ Solve for $f(x) = 0$

c) $7(x+1) - 4(2x+3) = 3(x+1)$

d) $\frac{t}{2} - \frac{t}{5} = 4$

e) $\frac{5x}{2} + \frac{3}{4}(x-4) = 0$

f) $\frac{10-4x}{4} = \frac{5x+6}{3} + 2$

g) $\frac{7x+3}{4x+1} = \frac{1}{3}$

Answers:

a) $x = {-14}/{3}$

b) $x = {21}/{5}$

c) $x = -2$

d) $t = {40}/{3}$

e) $x = {12}/{13}$

f) $x = {-9}/{16}$

g) $x = {-8}/{17}$