Existence and Uniqueness of Solutions

Basic Questions to be considered for an IVP $y' = f(t, y)$, $y(t_0) = y_0$

Existence: Under what precise conditions will the IVP have *at least* one solution?
Uniqueness: Under what precise conditions will the IVP have *at most* one solution?
Extension and Long-Term Behavior: How far ahead in the future and back into the past does a solution extend? How does a solution behave as t gets large?
Description: How can a solution and its behavior be described?

Existence and Uniqueness Theorem for Linear ODE's

If the function p and g are continuous on an open interval I containing the point t_0 , then there is a unique function $y(t)$ that satisfies the differential equation $y' + p(t)y = g(t)$ for each t in I, and that also satisfies the initial condition $y(t_0) = y_0$.

Example 1: Finding an interval for which a solution is guaranteed.

Determine an interval in which the solution of the IVP $t(t-4)y' + y = 0$, $y(2) = 1$ is certain to exist.

Existence and Uniqueness Theorem (General Case)
Suppose that the function $f(t, y)$ and $\partial f(t, y)/\partial t$ are continuous on a closed rectangle R of the ty-plane and that (t_0, y_0) is a point inside R. Then the IVP

$$y' = f(t, y),\ y(t_0) = y_0$$

has a solution $y(t)$ on some interval I containing t_0 in its interior (existence), but no more than one solution in R on any time interval containing t_0 (uniqueness).

Example 2: Identifying the points where EUT are satisfied

State where in the ty-plane the above theorem is satisfied for the equation:

a) $y' = (t^2 + y^2)^{3/2}$

b) $y' = (1 - t^2 - y^2)^{1/2}$

Example 3:

Consider the IVP $y' = 2y^{1/2}$, $y(t_0) = y_0$. $f(t, y) = 2y^{1/2}$ so that $df/dy = y^{-1/2}$ is not continuous at $y = 0$.

Example 4: An IVP with Infinitely Many Solutions and an IVP with no solutions

a) $ty' - y = t^2 \cos t$, $y(0) = 0$

b) $ty' - y = t^2 \cos t$, $y(0) = 1$