

Homogeneous Linear Equations

Definitions: A **second order linear differential equation** is a differential equation of the form $a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$. The equation is said to be **homogeneous** if $f(x) = 0$. In that case the equation has the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad (1.1)$$

Theorem: Let I be an open interval of real numbers, and let $y_1(x)$ and $y_2(x)$ be any two non-trivial solutions of the equation for x in I . Then the complete solution to the (1.1) is given by $y(x) = c_1y_1(x) + c_2y_2(x)$ where c_1, c_2 are arbitrary constants, provided the following conditions are satisfied

- $\frac{y_1(x)}{y_2(x)}$ is not constant (i.e. the two functions are linearly independent).
- $a(x), b(x)$, and $c(x)$ are continuous on I .
- $a(x) \neq 0$ for all $x \in I$.

Definition: A second-order differential equation is said to be a **second-order linear equation with constant coefficients** if it is of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$. The related homogeneous equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad (1.2)$$

is relatively easy to solve.

We might guess that a solution of equation (1.2) has the form $y = e^{rx}$. Differentiating twice and substituting into the equation yields

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Since $e^{rx} \neq 0$, we may divide both sides of the equation to yield

$$ar^2 + br + c = 0 \quad (1.3)$$

Definition: Equation (1.3) is called the **characteristic equation** of the differential equation (1.2).

Theorem: The general solution of the differential equation $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ depends upon the solution of the characteristic equation $ar^2 + br + c = 0$ in the following way:

- a) If the roots $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real and unequal, then the general solution of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

- b) If the roots $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real and equal ($\sqrt{b^2 - 4ac} = 0$), then the general solution of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_2 x}.$$

- c) If the roots $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are the complex conjugates $a \pm bi$ ($b^2 - 4ac < 0$), then the general solution of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ is

$$y(x) = e^{ax} (c_1 \cos bx + c_2 \sin bx).$$

Example 1: Solving second-order linear equations with constant coefficients

a) $y'' + 4y' - 5y = 0$

Answer: $y(x) = c_1 e^{-5x} + c_2 e^x$

b) $y'' + 4y' = 0$

Answer: $y(x) = c_1 e^{-4x} + c_2$

c) $y'' + 4y' + 4y = 0$

Answer: $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$
(Use reduction of order.)

d) $y'' + 4y' + 5y = 0$

Answer: $y(x) = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

e) $y'' + 4y' + 8y = 0$

Answer: $y(x) = c_1 e^{-2x} \cos 2x + c_2 e^{-2x} \sin 2x$

Second-Order Nonhomogeneous Linear Equations with Constant Coefficients

We now consider how to solve differential equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad (1.4)$$

where $f(x)$ is a continuous function.

Definition: Suppose we are given the differential equation $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$. The related equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad (1.5)$$

is called the **complementary equation**.

Theorem: The general solution of the differential equation $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$ can be written as

$$y(x) = y_c(x) + y_p(x)$$

where y_p is a particular solution of (1.4) and $y_c(x)$ is a solution of the complementary equation **Error! Reference source not found.**

The Method of Undetermined Coefficients

Suppose you are given a differential equation of the form $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$. We first solve the related homogeneous equation to find the complementary solution $y_c(x)$. We will then make a guess for the form of the particular solution y_p , substitute our guess into the differential equation and solve for the constants. The success of our method depends upon our ability to make a good guess. It should be noted that the method only works for a limited class of functions $f(x)$.

What to guess for a particular solution

$f(x) =$	Guess
$c_n x^n + \cdots + c_1 x + c_0$	$A_n x^n + \cdots + A_1 x + A_0$
$c e^{kx}$	$A e^{kx}$
$c \cos kx + d \sin kx$	$A \cos kx + B \sin kx$

If any term of the particular solution y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 as needed).

Example 2: Using the Method of Undetermined Coefficients

Solve the following equations.

a) $y'' + y' - 2y = x^2$

The related homogeneous equation is $y'' + y' - 2y = 0$ which has characteristic equation $r^2 + r - 2 = (r + 2)(r - 1) = 0$. This equation has two distinct solutions $r_1 = -2$ and $r_2 = 1$. The complementary solution of the differential equation is therefore of the form $y_c(x) = c_1e^{-2x} + c_2e^x$.

Since $f(x) = x^2$ is a second-degree polynomial, we guess that the particular solution is also a second-degree polynomial i.e. $y_p(x) = Ax^2 + Bx + C$. With this choice of particular solution,

we have $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Substituting into the original equation $y'' + y' - 2y = x^2$ gives

$$2A + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

This equation can be simplified to

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Since the coefficients of like powers of x must match on each side of the equation, we get the system of linear equations

$$-2A = 1, (2A - 2B) = 0, \text{ and } (2A + B - 2C) = 0.$$

The solution of this system of linear equations is $A = -\frac{1}{2}$, $B = -\frac{1}{2}$, and $C = -\frac{3}{4}$

The general solution of the equation is

$$y(x) = y_c(x) + y_p(x) = c_1e^{-2x} + c_2e^x - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

b) $y'' + 4y = e^{3x}$

Hint: Try $y_p(x) = Ae^{3x}$.

c) $y'' + y' - 2y = \sin x$

Hint: Try $y_p(x) = A \cos x + B \sin x$.

d) $y'' + y = \sin x$

Hint: Try $y_p(x) = Ax \cos x + Bx \sin x$.

Theorem: The **Principle of Superposition** says that if we have a differential equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f_1(x) + f_2(x)$$

and y_{p_1} is a particular solution of $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f_1(x)$

and y_{p_2} is a particular solution of $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f_2(x)$,

then $y_{p_1} + y_{p_2}$ is a particular solution of $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f_1(x) + f_2(x)$

Example 3: Using the Method of Undetermined Coefficients

Solve: $y'' - 4y = xe^x + \cos 2x$

Hint: Try $y_{p_1}(x) = (Ax + B)e^x$ and $y_{p_2}(x) = Cx \cos x + Dx \sin x$.