Solving Matrix Differential Equations

Steps for Solving a Matrix Differential Equation

1. Find the characteristic equation of $\mathbf{A}$, $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.
2. Find the eigenvalues of $\mathbf{A}$, which are the roots of the characteristic equation.
3. For each eigenvalues λ, find as many linearly independent eigenvectors as possible.
4. The solution of the matrix differential equation takes the form:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

Example 2: **Solving a system of two linear differential equations using eigenvalues**

Solve $\begin{aligned} \frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y \end{aligned}$.

The system of equations has the form $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$. In Example 1, we found the eigenvalues of this matrix to be $\lambda_1 = -1$ and $\lambda_2 = 4$. The corresponding eigenvectors were $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

The solution of the equation is therefore

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} -2c_1 e^{-t} + c_2 e^{4t} \\ 3c_1 e^{-t} + c_2 e^{4t} \end{bmatrix}.$$

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Mathematica:

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Example 3: Solving a system of two linear differential equations using eigenvalues

Solve $\begin{array}{l} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 4x + y \end{array}$.

Solution: The system of equations has the form $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. The eigenvalues of this matrix are given by the solutions of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Since $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ and $\lambda\mathbf{I} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, we have $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$. Setting the determinant of this matrix equal to zero gives: $\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(1-\lambda) - 4 = 0$.

Multiplying out, we have $\lambda^2 - 2\lambda - 3 = 0$. So, the two eigenvalues are $\lambda = -1, 3$ or $\lambda_1 = 3$ and $\lambda_2 = -1$.

Next, find the eigenvectors.

$\lambda_2 = -1$: With this choice of λ, the matrix $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

So that the matrix equation we must solve is $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation can be solved using the augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]$. This can be reduced to $\left[\begin{array}{cc|c} 1 & .5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so the solution is $v_1 + \frac{1}{2}v_2 = 0$ or $v_1 = -\frac{1}{2}v_2$. We choose $v_2 = -2$ so that we answer involves integers and we have $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\lambda_1 = 3$: With this choice of λ: $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$. So the system to be solved is $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The associated augmented matrix is $\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$.

This can be reduced to $\left[\begin{array}{cc|c} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so the solution is $v_1 - \frac{1}{2}v_2 = 0$ or $v_1 = \frac{1}{2}v_2$. We choose $v_2 = 2$ then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The solutions are then $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{y}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$.

We check the Wronskian to see if the solutions form a fundamental set:

$$W = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = e^{-t}\left(2e^{3t}\right) - \left(-2e^{-t}\right)e^{3t} = 4e^{2t} \neq 0.$$

Therefore the solutions are independent and the general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

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Example 4: Solving a system of two linear differential equations using eigenvalues

Solve $\begin{aligned} \frac{dx}{dt} &= -2x - 2y \\ \frac{dy}{dt} &= -x - 3y \end{aligned}$.

Solution: The system of equations has the form $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$. Then, we have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2-\lambda & -2 \\ -1 & -3-\lambda \end{bmatrix}.$$

Setting the determinant of this matrix equal to zero gives

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \left(2+\lambda\right)\left(3+\lambda\right) - 2 = 0.$$

The characteristic equation is

$$\lambda^2 + 5\lambda + 4 = 0.$$

The two eigenvalues are $\lambda = -1, -4$ or $\lambda_1 = -4$ and $\lambda_2 = -1$.

Next, use the eigenvalues to find the corresponding eigenvectors.

$\lambda_2 = -1$: With this choice of λ, the matrix $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2-\lambda & -2 \\ -1 & -3-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$. So we must solve the equation $\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{v} = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The corresponding augmented matrix is $\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -1 & -2 & 0 \end{array}\right]$. This can be reduced to $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so the solution is $v_1 + 2v_2 = 0$ or $v_1 = -2v_2$. We choose $v_2 = 1$ so that $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\lambda_1 = -4$: With this choice of λ: $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$. Then,

$$\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{v} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The associated augmented matrix is $\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -1 & 1 & 0 \end{array}\right]$. This can be reduced to $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so that $v_1 - v_2 = 0$ or $v_1 = v_2$. We choose $v_2 = 1$ then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The solutions are then $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}$ and $\mathbf{y}(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$.

Note: Check to see if the solutions form a fundamental set.

Tte general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}.$$

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Analysis of Example 4 Using Mathematica

Solving an IVP

To solve an initial value problem such as

$$\begin{array}{l} x' = -2x - 2y \\ y' = -x - 3y \end{array}, \quad x(0) = 1,\ y(0) = 0,$$ using *Mathematica* we proceed as follows.

Write the equations.

```
In[6]:=  e1 = ∂t x[t] == −2 ∗ x[t] − 2 ∗ y[t]

Out[6]=  x′[t] == -2 x[t] - 2 y[t]

In[7]:=  e2 = ∂t y[t] == −x[t] −3 ∗ y[t]

Out[7]=  y′[t] == -x[t] - 3 y[t]
```

Use the DSolve command as follows:

```
In[9]:=  DSolve[{e1, e2, x[0] == 1, y[0] == 0}, {x[t], y[t]}, t]
```

Out[9]= $$\left\{\left\{x[t] \to \frac{1}{3} e^{-4t} (1 + 2 e^{3t}),\ y[t] \to -\frac{1}{3} e^{-4t} (-1 + e^{3t})\right\}\right\}$$

Drawing Solutions in the Phase Plane

To draw a few solution for a system such as $\begin{array}{l} x' = -2x - 2y \\ y' = -x - 3y \end{array}$ for initial conditions x (0),

y (0) between − 3 and 3,

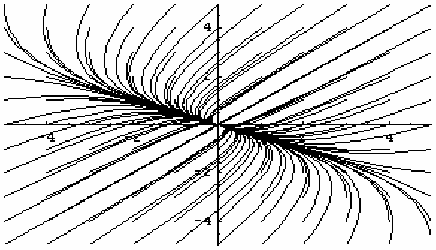
we do the following:

First write the equations :

```
In[10]:=  e1 = ∂t x[t] == −2 ∗ x[t] − 2 ∗ y[t]

Out[10]=  x′[t] == -2 x[t] - 2 y[t]

In[11]:=  e2 = ∂t y[t] == −x[t] −3 ∗ y[t]

Out[11]=  y′[t] == -x[t] - 3 y[t]
```

Then then use the following program:

```
In[18]:=  numsol[t0_, a_, b_, t1_] := ({x[t], y[t]} /.
                  First[NDSolve[{e1, e2, x[0] == a, y[0] == b}, {x[t], y[t]}, {t, 0, t1}]]) /.
             t → t0
          flow[t1_] := ParametricPlot[Evaluate[Flatten[Table[numsol[t, a, b, t1], {a, −5, 5}, {b, −5, 5}], 1]],
             {t, 0, t1}, PlotRange → {{−5, 5}, {−5, 5}}]
          flow[6]
```



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Out[20]=  - Graphics -
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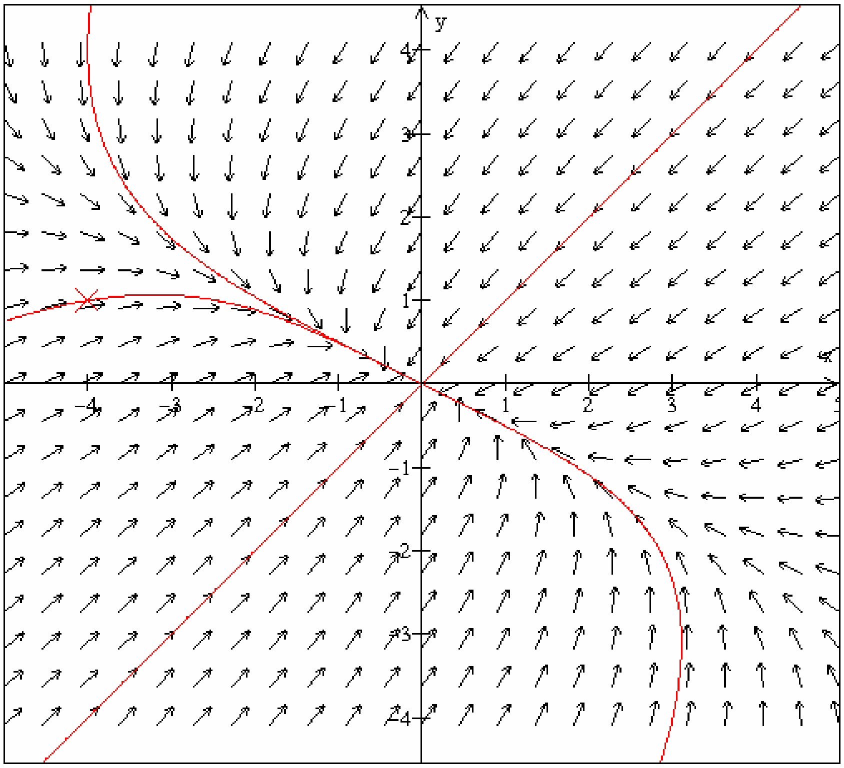
Equilibrium Solutions of Linear Systems

A linear system of two differential equations of the form $\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$ has $(0,0)$ as an equilibrium solution. The type of equilibrium point depends on the eigenvalues λ_1, λ_2 of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For now we consider the cases where the system has nonzero, distinct, real eigenvalues.

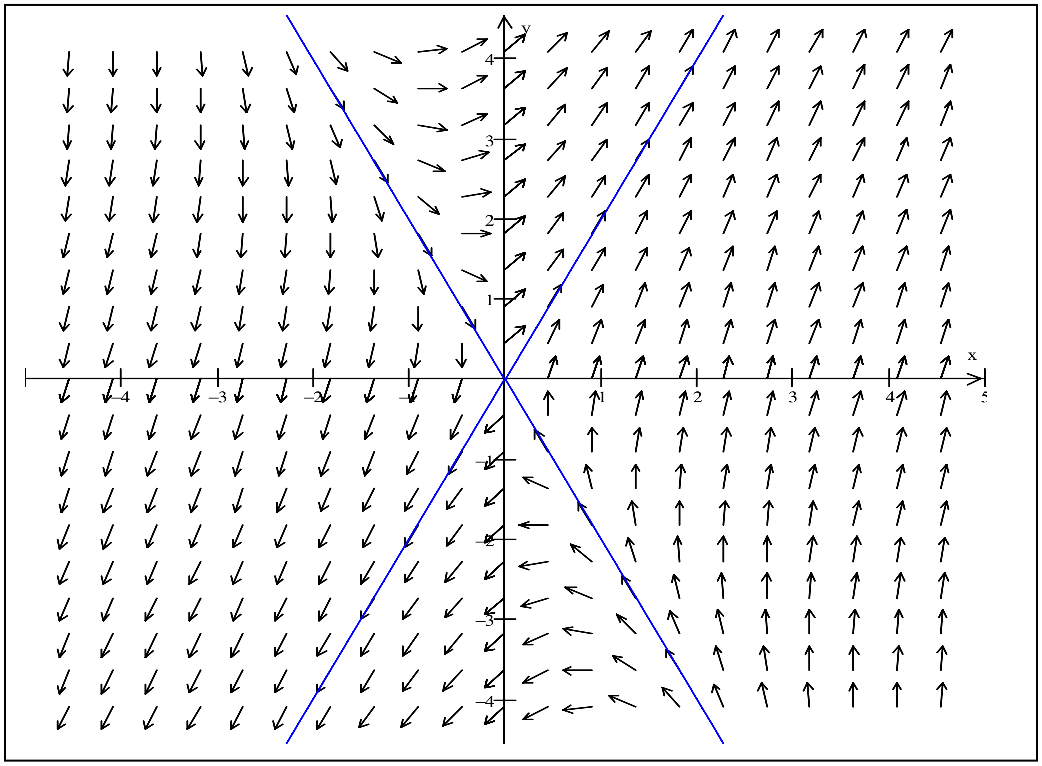
Three Types of Equilibrium Points
If a linear system of two differential equations has two nonzero, distinct, real eigenvalues λ_1, λ_2, then:

- If $\lambda_1 < 0 < \lambda_2$, then the origin is a **saddle**. There are two lines in the phase plane that correspond to straight-line solutions.
- If $\lambda_1 < \lambda_2 < 0$, then the origin is a **sink** (stable node). All solutions tend to $(0,0)$ as $t \rightarrow \infty$ and most go to zero in the direction of the λ_2-eigenvectors.
- If $0 < \lambda_2 < \lambda_1$, then the origin is a **source** (unstable node). All solutions except the equilibrium solution tend to infinity as $t \rightarrow \infty$ and most solutions leave the origin in the direction of the λ_2-eigenvectors.



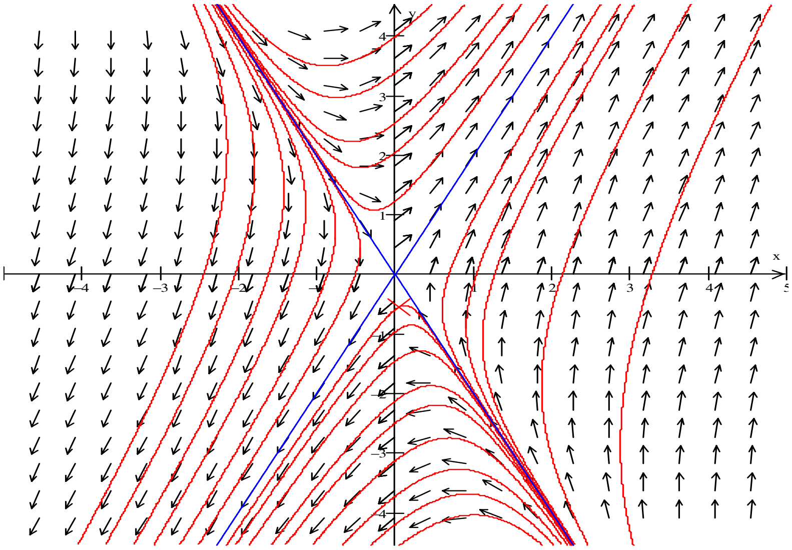
The sink (stable node) of Example 4

Graphical Analysis of Example 3



Saddle point at (0,0) from Example 3

The two lines correspond to the two lines of eigenvectors. The line $y = 2x$ corresponds to the eigenvector vector $\begin{bmatrix}1\\2\end{bmatrix}$ and the line $y = -2x$ corresponds to the eigenvector vector $\begin{bmatrix}1\\-2\end{bmatrix}$. We can also see that since, $x(t) = c_1e^{3t}$ and $y(t) = 2c_1e^{3t}$, we have $y(x) = 2\left(c_1e^{3t}\right) = 2x$.



Example 5: Solving a system of three linear differential equations using eigenvalues

Solve the system of three linear differential equations $$\begin{aligned} \frac{dx}{dt} &= y+z \\ \frac{dy}{dt} &= x+z \\ \frac{dz}{dt} &= x+y \end{aligned}.$$

Solution: First, write the system as a matrix equation: $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$.

Solutions involving Complex Eigenvalues

Example 6: Solving a system with complex eigenvalues

Solve the system $\begin{aligned}\frac{dx}{dt} &= -2x - 3y \\ \frac{dy}{dt} &= 3x - 2y\end{aligned}$

Solution: The characteristic equation of this system is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-2-\lambda)(-2-\lambda) + 9 = 0$$

which simplifies to $\lambda^2 + 4\lambda + 13 = 0$. The eigenvalues are $\lambda_1 = -2+3i$ and $\lambda_2 = -2-3i$. To find the eigenvector corresponding to $\lambda_1 = -2+3i$ we substitute into the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ to get $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We solve the system of equations $\begin{aligned} -3iv_1 - 3v_2 &= 0 \\ 3v_1 - 3iv_2 &= 0 \end{aligned}$ by using the bottom equation to get $v_1 = iv_2$ (which also satisfies the first equation). We choose $\begin{bmatrix} i \\ 1 \end{bmatrix}$ as the eigenvector.

In order to finish, we need the following result from precalculus:

Euler's Formula: $e^{ib} = \cos b + i\sin b$ $\qquad$ $e^{a+ib} = e^{a}(\cos b + i\sin b)$

Continuing with our example, we see that $e^{(-2+3i)t} = e^{-2t}\left[(\cos 3t) + i\sin(3t)\right]$. It follows that the solution has the form:

$$\mathbf{x}(t) = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{-2t}\left[(\cos 3t) + i\sin(3t)\right]$$

$$= \begin{bmatrix} ie^{-2t}\left[(\cos 3t) + i\sin(3t)\right] \\ e^{-2t}\left[(\cos 3t) + i\sin(3t)\right] \end{bmatrix} = \begin{bmatrix} ie^{-2t}\cos(3t) - e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) + ie^{-2t}\sin(3t) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix} + i\begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$$

The two pieces are the real and imaginary parts of the vector function $\mathbf{x}(t)$.

Theorem: If $\mathbf{x}(t)$ is a complex-valued solution to a linear system $\mathbf{x}'(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}(t)$ where the coefficient matrix has all real entries, and that $\mathbf{x}(t)$ can be written as the sum of real and imaginary parts as $\mathbf{x}(t) = \mathbf{x}(t)_{re} + \mathbf{x}(t)_{im}$ where both $\mathbf{x}(t)_{re}, \mathbf{x}(t)_{im}$ are real-valued functions of *t*. Then $\mathbf{x}(t)_{re}$ and $\mathbf{x}(t)_{im}$ are both solutions of the original system.

Example 6: Conclusion

The functions $\mathbf{x}_{re}(t) = \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix}$ and $\mathbf{x}_{im}(t) = \begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$ are both solutions of the system. They are independent since their initial values $\mathbf{x}_{re}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{x}_{im}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are independent. Therefore, the general solution of the system is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$$

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Example 7: Solving a system with complex eigenvalues

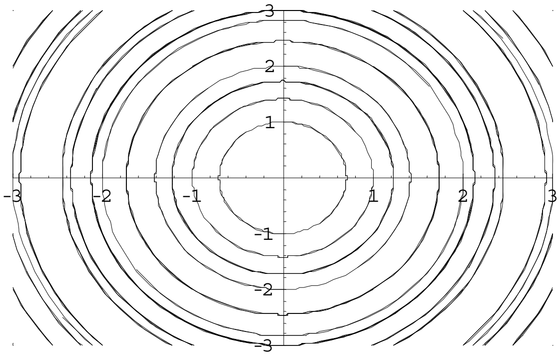
Solve the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -3 & 2 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 8: Solving a system with complex eigenvalues

A harmonic oscillator can be modeled by the second-order equation $my'' + by' + ky = 0$. This is considered undamped if $b = 0$. Choose $b = 0$ and choose the mass to be $m = 1$ and the spring constant to be $k = 2$. Then, the equation can written as a system by letting $x_1 = y$ and $x_2 = x_1{}'$, so that $\begin{aligned} x_1{}' &= x_2 \\ x_2{}'' &= -2x_1 \end{aligned}$ or

$$\mathbf{X}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{X}(t).$$

Sorry but I don't have this one typed up yet!!



Deficient Eigenvalues

Let the system of linear differential equations with constant coefficients be given by $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$. Suppose λ is the only eigenvalue. We seek a solution of the form:

$$\mathbf{x}(t) = \left(\mathbf{v}_1 t + \mathbf{v}_2\right)e^{\lambda t} .$$

Here $\mathbf{v}_1, \mathbf{v}_2$ are constant column matrices. Differentiating the proposed solution gives

$$\mathbf{x}'(t) = \left[\left(\mathbf{v}_1 t + \mathbf{v}_2\right)e^{\lambda t}\right]' = \mathbf{v}_1 e^{\lambda t} + \lambda\left(\mathbf{v}_1 t + \mathbf{v}_2\right)e^{\lambda t} .$$

Substituting into the differential equation gives

$$\mathbf{v}_1 e^{\lambda t} + \lambda\left(\mathbf{v}_1 t + \mathbf{v}_2\right)e^{\lambda t} = \mathbf{A}\left(\mathbf{v}_1 t + \mathbf{v}_2\right)e^{\lambda t} .$$

Dividing out the exponential, we have

$$\mathbf{v}_1 + \lambda\left(\mathbf{v}_1 t + \mathbf{v}_2\right) = \mathbf{A}\left(\mathbf{v}_1 t + \mathbf{v}_2\right)$$

or

$$\mathbf{v}_1 + \lambda \mathbf{v}_1 t + \lambda \mathbf{v}_2 = \mathbf{A}\mathbf{v}_1 t + \mathbf{A}\mathbf{v}_2 \ .$$

Equating coefficients gives the system of equations

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \lambda \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \mathbf{v}_1 + \lambda \mathbf{v}_2 \end{aligned}$$

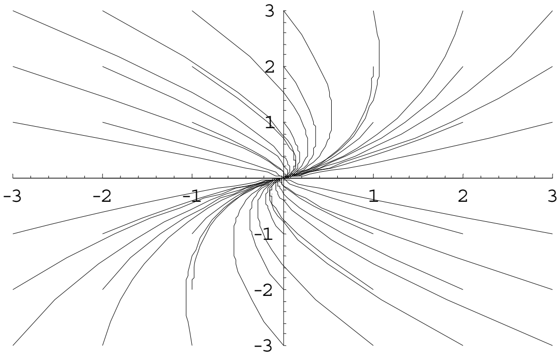
Then $\mathbf{v}_1$ is the eigenvector of $\mathbf{A}$ associated with λ and $\mathbf{v}_2$ is the solution of the second equation. The first equation is how we got the original eigenvalue and eigenvector. Solving the second equation is equivalent to solving

$$\left(\mathbf{A} - \lambda I\right)\mathbf{v}_2 = \mathbf{v}_1 .$$

Example 9: Solving a system with repeated eigenvalues

Solve the linear system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$.

Solution: First note that the phase plot shows that there is only one straight line of solutions which indicates that the system has a single eigenvalue.



The eigenvalue of $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ is given by the solution of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 = 0\,.$$

So the system $\mathbf{x}'(t) = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t)$ has $\lambda = -2$ as its only eigenvalue. The associated eigenvector is found from $\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1$ or $(\mathbf{A} + 2\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Since $(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we must solve the system $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It follows that $v_2 = 0$. The eigenvector can be chosen to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The first solution is given by $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$

To find a second eigenvector and a second solution, we must solve the system $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$ which is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$ or in this case:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This results in $v_2 = 1$. So we can choose the second eigenvector $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then the second solution will be of the form: $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$.

The general solution will then be of the form:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \right)$$

or $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} c_2 \\ 0 \end{bmatrix} te^{-2t}$

Theorem: Suppose $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is a linear system in which the 2×2 matrix $\mathbf{A}$ has a repeated real eigenvalue λ but only one line of eigenvectors. Then the general solution has the form

$$\mathbf{x}(t) = \mathbf{V}_0 e^{\lambda t} + \mathbf{V}_1 te^{\lambda t}$$

where $\mathbf{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is an arbitrary initial condition and V_1 is determined from $\mathbf{V}_0$ by $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$. If $V_1 = 0$, then $\mathbf{V}_0$ is an eigenvector and $\mathbf{x}(t)$ is a straight-line solution.

Example 9: Conclusion

The system $\mathbf{x}'(t) = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t)$ has $\lambda = -2$ as its only eigenvalues. Let $\mathbf{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be an arbitrary initial condition. Then

$\mathbf{V}_1 = (\mathbf{A} + 2\mathbf{I})\mathbf{V}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \end{bmatrix}$. So the general solution

is $\mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{-2t} + \begin{bmatrix} y_0 \\ 0 \end{bmatrix} te^{-2t}$.

Example 10: Solving a system with repeated eigenvalues

Solve the linear system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

Solution: The eigenvalues are solutions of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = 0\,.$$

The equation $(1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$ has $\lambda = 2$ as its only solution. So $\mathbf{A}$ has $\lambda = 2$ as its only eigenvalue. The associated eigenvector is found from $\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1$ or $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Since $(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, we must solve the system $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It follows that $v_1 = -v_2$. The eigenvector can be chosen to be $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The first solution is given by $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

To find a second eigenvector and a second solution, we must solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$ or in this case: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This system has the augmented matrix $\left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]$ which reduces to $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$. It follows that $v_1 = -v_2 - 1$. We choose $v_1 = 0$ then the eigenvector is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then the second solution will be of the form: $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ The general solution will then be of the form:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}\right)$$

or $\mathbf{x}(t) = \begin{bmatrix} c_1 e^{2t} + c_2 te^{2t} \\ -(c_1 + c_2)e^{2t} - c_2 te^{2t} \end{bmatrix}$.

Theorem: Suppose $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is a linear system in which the 2×2 matrix $\mathbf{A}$ has a repeated real eigenvalues λ but only one line of eigenvectors. Then the general solution has the form

$$\mathbf{x}(t) = \mathbf{V}_0 e^{\lambda t} + \mathbf{V}_1 t e^{\lambda t}$$

where $\mathbf{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is an arbitrary initial condition and V_1 is determined from $\mathbf{V}_0$ by $\mathbf{V}_1 = (\mathbf{A} - \lambda \mathbf{I})\mathbf{V}_0$. If $V_1 = 0$, then $\mathbf{V}_0$ is an eigenvector and $\mathbf{x}(t)$ is a straight-line solution.

Example 10: Conclusion

The system $\mathbf{x}'(t) = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t)$ has $\lambda = -2$ as its only eigenvalues. Let $\mathbf{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be an arbitrary initial condition. Then

$\mathbf{V}_1 = (\mathbf{A} + 2\mathbf{I})\mathbf{V}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \end{bmatrix}$. So the general solution

is $\mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{-2t} + \begin{bmatrix} y_0 \\ 0 \end{bmatrix} t e^{-2t}$.

Example 10: A Harmonic Oscillator

Consider the harmonic oscillator modeled by the second-order equation $y'' + 2\sqrt{2}y' + 2y = 0$, with mass $m = 1$ and spring constant $k = 2$ and damping coefficient $b = 2\sqrt{2}$. The equation can written as a system by letting $x_1 = y$ and $x_2 = {x_1}'$, so that $\begin{matrix} {x_1}' = x_2 \\ {x_2}' = -2x_1 - 2x_2 \end{matrix}$ or $\mathbf{X}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{bmatrix} \mathbf{X}(t)$.