Oscillations

Suppose we have a spring of natural length L and spring constant k, with its upper end fastened to a rigid support. We hang a mass m from the spring. The weight of the mass stretches the spring to a length $L+s$ when allowed to come to rest in its new equilibrium position. By Hooke's law, the tension in the spring is ks. The force of gravity pulling down the mass is mg. Equilibrium requires that

$$ks = mg$$

If we pull down the mass an additional amount x_0 and release it, the forces acting on the mass are $+mg$ and $-k(s+x)$ where x denotes the displacement of the mass from equilibrium at time t. By Newton's second law, the sum of the forces satisfies

$$m\frac{d^2x}{dt^2} = mg - ks - kx$$

since $mg = ks$, this simplifies to

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.1}$$

In addition to satisfying this equation, the mass must also satisfy the initial conditions

$$x = x_0 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = 0\,.$$

If we divide both sides of (1.1) by m and let $\omega = \sqrt{\frac{k}{m}}$ the equation becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This equation has the solution

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)\,. \tag{1.2}$$

Using the initial conditions listed above, we obtain the solution

$$x(t) = x_0\cos(\omega t)$$

which represents a simple harmonic motion. The two terms of the general solution (1.2)can be combined into a single term using trigonometric identities into

$$x(t) = C\cos(\omega t - \phi)$$

This represents a simple harmonic motion of amplitude C and period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$. The angle ϕ is the **phase angle** of the motion.

Important note: In order to put the equation in this last form, we must have

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Equal to $$x(t) = C\cos(\phi)\cos(\omega t) + C\sin(\phi)\sin(\omega t) = C\cos(\omega t - \phi)$$

The coefficients are related by the equations

$c_1 = C\cos\phi$, $c_2 = C\sin\phi$, from which it follows that

$C = \sqrt{c_1^2 + c_2^2}$, and $\tan\phi = \dfrac{c_2}{c_1}$.

Example 1: Undamped Spring

A mass weighing 10 lbs stretches a spring 2 inches. Suppose the mass is displaced another 2 inches in the positive direction then set in motion with an upward velocity of 1 ft/sec, determine the position of the mass at any later time t.

Solution: The spring constant $k = 10lb/2in = 60lb/ft$, and the mass is $m = w/g = 10/32\text{lb-sec}^2/\text{ft}$. The resulting initial value problem is therefore,

$$\frac{d^2x}{dt^2} + 192x = 0\,,\; x(0) = 1/6\,\mathbf{ft},\; x'(0) = -1ft/\sec$$

Damped Oscillation

Suppose the mass is slowed by a frictional force that is proportional to velocity, $c\left(\frac{dx}{dt}\right)$, where c is a positive constant. Then the equation becomes

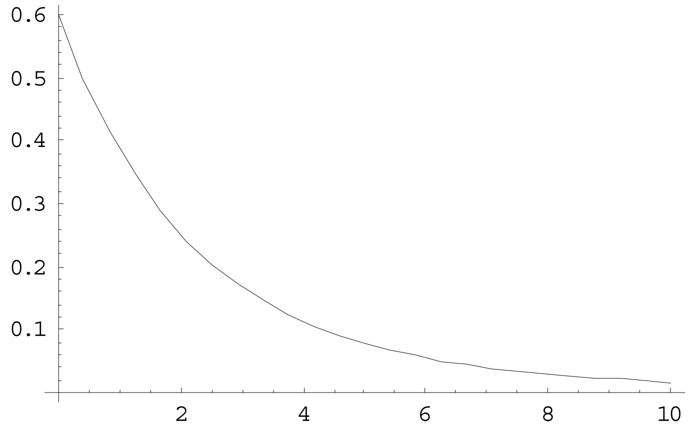
$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$$

where $2b = \frac{c}{m}$.

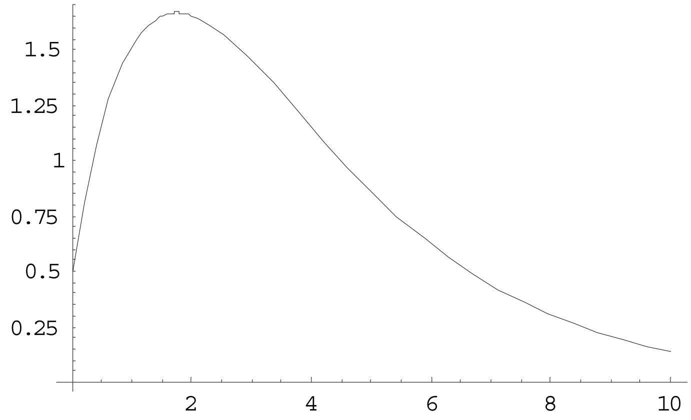
The roots of the characteristic equation $\lambda^2 + 2b\lambda + \omega^2 = 0$ are $\lambda = -b \pm \sqrt{b^2 - \omega^2}$. The mass behaves in three distinct ways depending upon the relative sizes of b and ω.

1. **Critical Damping**: $(b = \omega)$ In this case, the roots are equal and the solution is $x(t) = (c_1 t + c_2)e^{-\omega t}$
2. **Overcritical Damping**: Overdamping: $(b > \omega)$ In this case, the roots are distinct real numbers and the solution is $x(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ It follows that $x(t) \to 0$ as $t \to \infty$. Oscillations do not occur.
3. **Undercritical Damping**: Underdamping: $(0 < b < \omega)$ In this case, the roots are complex conjugate numbers $r = -b \pm \alpha i$ and the solution is $x(t) = e^{-bt}(c_1 \cos \alpha t + c_2 \sin \alpha t)$ It follows that $x(t) \to 0$ as $t \to \infty$. Oscillations do occur but they die out as time increases. This equation may also be written in the equivalent form $x(t) = Ce^{-bt} \cos(\alpha t - \phi)$. Notice that the amplitude is not constant but is given by Ce^{-bt}. $b = \frac{c}{2m}$ appears in the exponential **damping factor.** It follows that the larger b is the faster the oscillations become unnoticeable. The period $T = \frac{2\pi}{a} = \frac{2\pi}{\sqrt{\omega^2 - b^2}}$ is longer than the period $T_0 = \frac{2\pi}{\omega}$ in the friction-free system. The motion is slower.

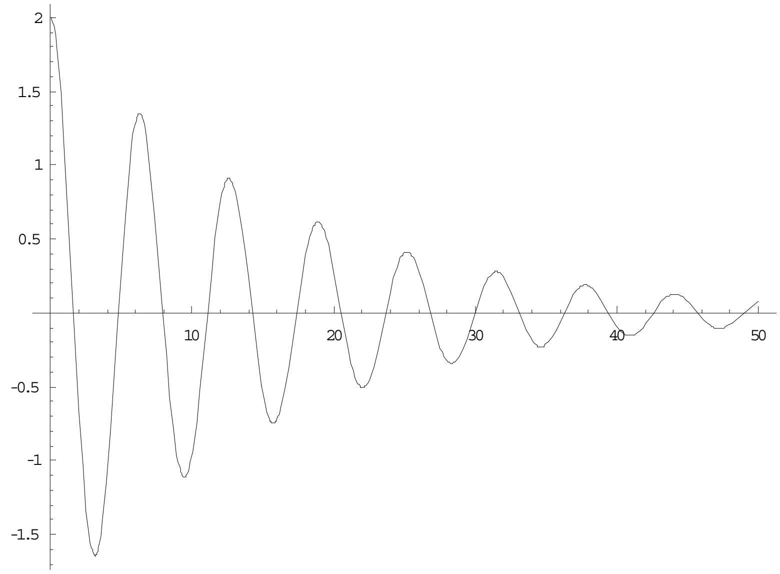
Overcritical Damping:



Critical Damping



Undercritical Damping:



Forced Undamped Oscillations

Assume that the system is undamped (internal friction is negligible) but that a periodic external force, say $f(t) = a\cos\omega t$, acts vertically on the spring mass system. Then the equation of the forced undamped oscillator is

$$\frac{d^2x}{dt^2} + \omega_0{}^2 x = a\cos(\omega t)$$

Where we use $\omega_0 = \sqrt{\frac{k}{m}}$ to represent the natural frequency of the spring. There are two cases to consider: $\omega \neq \omega_0$ and $\omega = \omega_0$.

Case 1: Beats $(\omega \neq \omega_0)$

In this case the general solution is $x(t) = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t) + \frac{a\cos(\omega t)}{\omega_0^2 - \omega^2}$.

The simplest motion of this system occurs when the spring-mass system is at rest and only the external force acts on it. Then the initial conditions are $x(0) = x'(0) = 0$, which means the constants are $c_1 = 0$ and $c_2 = \frac{-a}{\omega_0^2 - \omega^2}$ and the particular solution is $x(t) = \frac{a}{\omega_0^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right]$. This can be simplified to

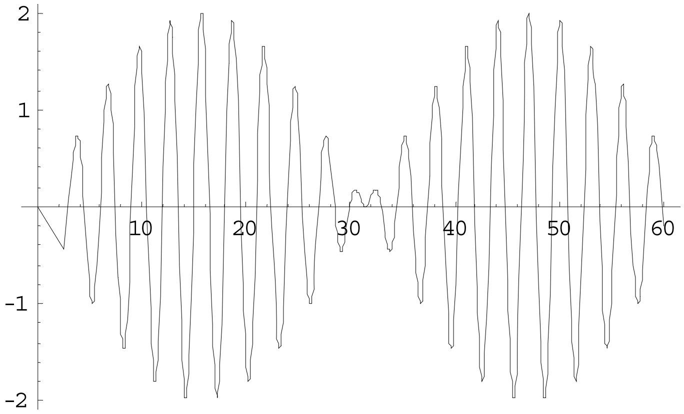
$$x(t) = A(t)\left[\sin\left(\frac{\omega_0 + \omega t}{2}\right)\right], \qquad (1.1)$$

where $A(t) = \frac{2a}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega_0 - \omega}{2}t\right)$ can be interpreted as a variable amplitude. When ω is close to ω_0 a solution is referred to as a **beat**.

Example 2: **An Undamped Forced oscillator with Beats**

Solve and graph the solution to $\frac{d^2x}{dt^2} + 4.81x = 0.81\cos(2t)$ with the initial conditions $x(0) = x'(0) = 0$.

Solution: According to (1.1) the solution will be of the form $x(t) = 2\sin\left(\frac{t}{10}\right)\sin\left(\frac{21t}{10}\right)$. The graph is shown below.



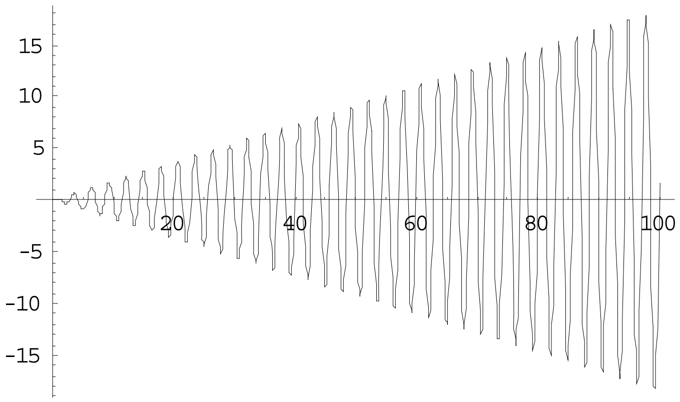
Case 2: **Resonance** $(\omega = \omega_0)$

In this case, the equation is $\frac{d^2x}{dt^2} + \omega_0^2 x = a\cos(\omega_0 t)$ which has the general solution $x(t) = \left(c_1 + \frac{at}{2\omega_0}\right)\sin(\omega_0 t) + c_2\cos(\omega_0 t)$. This means that as $t \to \infty$, x becomes unbounded and the amplitude, defined as $A(t) = \sqrt{\left(c_1 + \frac{at}{2\omega_0}\right)^2 + c_2^{\,2}}$, tends to infinity. This is called **resonance**.

Example 2: **An Undamped Forced Oscillator with Resonance**

Solve and graph the solution to $\frac{d^2x}{dt^2} + 4.81x = 0.81\cos(2.2t)$ with the initial conditions $x(0) = x'(0) = 0$.

Solution: The solution is $x(t) = \frac{.81t}{4.4}\sin(2.2t)$. The graph is shown below.


15
10
5
-5
-10
-15
20
40
60
80
100

Beats

a) Use the method of undetermined coefficients to show that the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = a\cos(\omega t)$$ has the general solution

$$x(t) = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t) + \frac{a\cos(\omega t)}{\omega_0^2 - \omega^2}.$$

b) Use the formula $\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$ show that the initial conditions $x(0) = x'(0) = 0$ lead to the particular solution $x(t) = A(t)\left[\sin\left(\frac{\omega_0 + \omega t}{2}\right)\right]$, where $A(t) = \frac{2a}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega_0 - \omega}{2}t\right)$.

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Resonance

Use the method of undetermined coefficients to show that in the case of resonance the equation is $\frac{d^2x}{dt^2} + \omega_0^2 x = a\cos(\omega_0 t)$ has the general solution $x(t) = \left(c_1 + \frac{at}{2\omega_0}\right)\sin(\omega_0 t) + c_2\cos(\omega_0 t)$