Differential Equations

Test 1

Summer 2002

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Instructions: Do 10 of the following 12 problems. You may do one problem for 5 points extra credit. Extra credit will only be awarded for a complete and correct solution.

Name ______________________________ - Test 1 Differential Equations Mike Huff

1. Verify that the function $f(x) = x^2 \ln x$ is a solution of the differential equation

$$x^2 y'' - 3xy' + 4y = 0$$

$y = x^2 \ln x$

$y' = x^2 \frac{1}{x} + 2x\ln x = x + 2x\ln x$

$y'' = 1 + 2x(\frac{1}{x}) + 2\ln x = 3 + 2\ln x$

$x^2(3+2\ln x) - 3x(x+2x\ln x) + 4x^2 \ln x$

$= \cancel{3x^2} + 2x^2\ln x - \cancel{3x^2} - 6x^2\ln x + 4x^2\ln x$

$= 0$ ✓

2. A ball with a mass of 0.25 kg is thrown upward with an initial velocity of 20 m/sec f from the roof of a building 30 m high. Neglect air resistance.

a) Find the maximum height above the ground the ball reaches.
b) Assuming the ball misses the building on the way down, find the time it takes for the ball to hit the ground.

$V(0) = 20\,m/s$

$X(0) = 30\,m$

$\frac{dv}{dt} = -9.8\ m/s^2$

$v = -9.8t + v_0$

$v(t) = -9.8t + 20$

a) max height occurs when $v(t) = 0$

$0 = -9.8t + 20 \Rightarrow t \approx 2.04s$

now use (b)!

$x(5.24) \approx 50.4\ m$

b) $x(t) = \int v(t)\,dt = \int(-9.8t + 20)\,dt = -4.9t^2 + 20t + X_0$

$X(0) = 30 \Rightarrow X_0 = 30$

$X(t) = -4.9t^2 + 20t + 30$

At the ground: $X(t) = 0$: $0 = -4.9t^2 + 20t + 30$

$t \approx 5.24s$

3. Solve the initial value problem: $\frac{dy}{dx} = xe^{-x}$, $y(0) = 1$.

$$\frac{dy}{dx} = xe^{-x}$$

$$\int dy = \int xe^{-x}\,dx$$

$$y(x) = (-x-1)e^{-x} + C$$

Since $y(0) = 1$

$$1 = (-0-1)e^{-0} + C$$

$$1 = -1 + C \Rightarrow C = 2$$

$$y(x) = 2 - xe^{-x} - e^{-x}$$

$$= 2 - e^{-x}(x+1)$$

4. Find the position function $x(t)$ of a moving particle with the acceleration function $a(t) = \frac{1}{(t+1)^3}$, initial position $x(0) = 0$ and initial velocity $v(0) = 0$.

$$a(t) = \frac{dv}{dt} = \frac{1}{(t+1)^3}$$

$$v(t) = \int a(t)\,dt = \int \frac{dv}{dt}\,dt = \int \frac{1}{(t+1)^3}\,dt = \frac{-1}{2(t+1)^2} + C$$

$\therefore\ v(t) = \frac{-1}{2(t+1)^2} + C \qquad v(0) = 0 \Rightarrow \qquad 0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

So $v(t) = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$

$$x(t) = \int v(t)\,dt = \int\left(-\frac{1}{2}(t+1)^{-2} + \frac{1}{2}\right)dt = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}t + C$$

$x(0) = 0 \Rightarrow \quad C = \frac{-1}{2} \qquad\qquad x(t) = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}t - \frac{1}{2}$

5. Find the general solution of the differential equation $y' = \dfrac{xy + x - y - x^2}{xy - y^2}$.

$$y' = \frac{xy + x - y - x^2}{xy - y^2}$$

$$y' = \frac{x - y - x^2 + xy}{y(x-y)}$$

$$y' = \frac{(x-y) - x(x-y)}{y(x-y)}$$

$$y' = \frac{1-x}{y}$$

$$\int y\,dy = \int (1-x)\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c$$

$$\cancel{y = \pm\sqrt{x - \frac{x^2}{2} + c}}$$

$$y^2 = 2x - x^2 + c$$

$$y = \pm\sqrt{2x - x^2 + c}$$

6. Find the general solution of the differential equation $(x^2+1)y' = 2xy + x^3$.

$$(x^2+1)y' - 2xy = x^3$$

$$y' - \frac{2x}{x^2+1}y = \frac{x^3}{x^2+1}$$

$$\rho = e^{\int \frac{-2x}{x^2+1}dx} = e^{-\ln(x^2+1)} = \frac{1}{x^2+1}$$

$$\frac{1}{x^2+1}y' - \frac{2x}{(x^2+1)^2}y = \frac{x^3}{(x^2+1)^2}$$

$$\left(\frac{1}{x^2+1}y\right)' = \frac{x^3}{(x^2+1)^2}$$

$$\frac{1}{x^2+1}y = \int \frac{x^3}{(x^2+1)^2}dx$$

Aside: $\displaystyle\int \frac{x^3}{(x^2+1)^2}dx$ $\quad u = x^2+1$, $du = 2x\,dx$, $x^2 = u-1$

$$= \frac{1}{2}\int \frac{u-1}{u^2}du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right)du = \frac{1}{2}\left(\ln|u| + \frac{1}{u}\right)$$

$$\frac{1}{x^2+1}y = \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$$

$$y = \frac{1}{2}(x^2+1)\ln(x^2+1) + \frac{1}{2} + C(x^2+1)$$

7. Find a particular solution of the differential equation $y' = (5-2y)y$, $y(0) = 1$.

$$\frac{1}{(5-2y)} \cdot \frac{1}{y} y' = 1$$

$$\frac{1}{(5-2y)y} = \frac{A}{5-2y} + \frac{B}{y} = \frac{2}{5} \cdot \frac{1}{5-2y} + \frac{1}{5}\frac{1}{y}$$

$$1 = Ay + B(5-2y)$$

$$y=0 \Rightarrow B = 1/5$$

$$y = 5/2 \Rightarrow \frac{5}{2}A = 1 \Rightarrow A = 2/5$$

$$\frac{2}{5}\int \frac{dy}{5-2y} + \frac{1}{5}\int \frac{dy}{y} = \int 1 \cdot dx$$

$$-\frac{1}{5}\ln|5-2y| + \frac{1}{5}\ln|y| = x + C$$

$$\ln|5-2y| - \ln|y| = -5x + c$$

$$\ln\left|\frac{5-2y}{y}\right| = -5x + c$$

$$\frac{5-2y}{y} = c\,e^{-5x}$$

$$5 - 2y = cye^{-5x}$$

$$cye^{-5x} + 2y = 5$$

$$y = \frac{5}{2 + ce^{-5x}}$$

$$y(0) = 1 \Rightarrow 1 = \frac{5}{2+c} \Rightarrow c = 3$$

$$\therefore y(x) = \frac{5}{2+3e^{-5x}}$$

8. A pitcher of iced tea initially $25^\circ C$ is to be cooled by placing it on the front porch, where the temperature is $0^\circ C$. Suppose that the temperature of the iced tea has dropped to $15^\circ C$ after 20 min. When will it be at $5^\circ C$?

$$\frac{dT}{dt} = k(A-T) \qquad A = 0$$

$$\frac{dT}{dt} = -kT$$

$$\frac{1}{T}\frac{dT}{dt} = -k$$

$$\ln|T| = -kt + c$$

$$T = Ce^{-kt}$$

$$T(0) = 25^\circ C$$

$$\Rightarrow T(t) = 25e^{-kt}$$

$$T(20) = 15^\circ C$$

$$15 = 25\,e^{-20k}$$

$$.6 = e^{-20k}$$

$$k = \frac{\ln .6}{-20} \approx 0.02554$$

$$T(t) = 25e^{-.02554t}$$

$$5 = 25e^{-.02554t}$$

$$.2 = e^{-.02554t}$$

$$t = \frac{\ln .2}{-.02554} \approx 63.016 \text{ min}$$

9. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$.

$$\frac{2y^3-y}{y^5}\,dy = \frac{x-1}{x^2}\,dx$$

$$\int\left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy = \int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$-\frac{2}{y}+\frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$$

10. Write a differential equation that is a mathematical model of the situation described:

a) The time rate of change of population P is proportional to the square root of P.

$$\frac{dP}{dt} = k\sqrt{P}$$

b) The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.

$$\frac{dv}{dt} = kv^2$$

c) In a city having a fixed population P of persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not heard the rumor.

$$\frac{dN}{dt} = k(P-N)$$

d) Solve one of the equations you have written above.

a)

$$\frac{dP}{dt} = k\sqrt{P}$$

$$\int P^{-1/2}\,dP = \int k\,dt$$

$$2P^{1/2} = kt + C$$

$$P^{1/2} = \frac{1}{2}kt + C$$

$$P(t) = \left(\frac{1}{2}kt + C\right)^2$$

b)

$$\frac{dv}{dt} = kv^2$$

$$\int v^{-2}\,dv = \int k\,dt$$

$$-v^{-1} = kt + C$$

$$v^{-1} = -kt + C$$

$$V(t) = \frac{1}{C - kt}$$

c)

$$\frac{dN}{dt} = k(P-N)$$

$$\int \frac{1}{P-N}\,dN = \int k\,dt$$

$$-\ln|P-N| = kt + C$$

$$\ln|P-N| = -kt + C$$

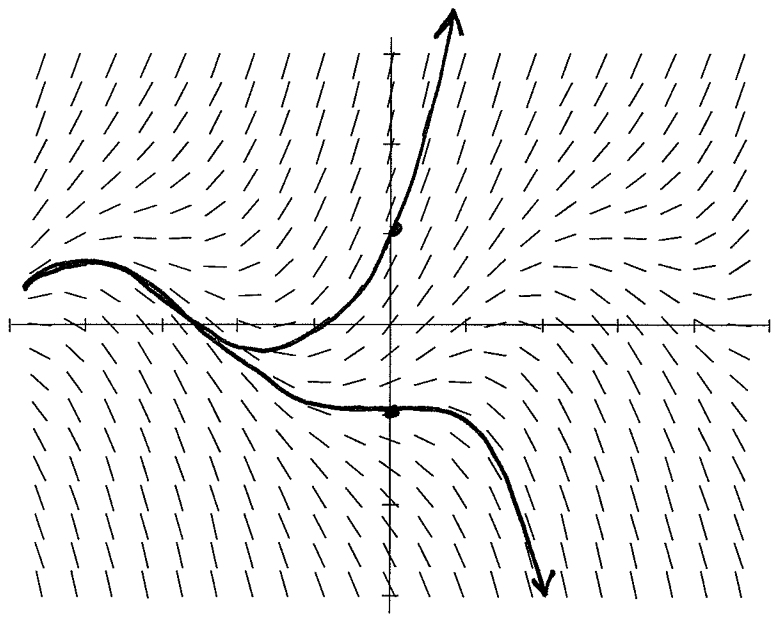
$$|P-N| = Ce^{-kt}$$

$$P-N = Ce^{-kt}, \quad N \leq P$$

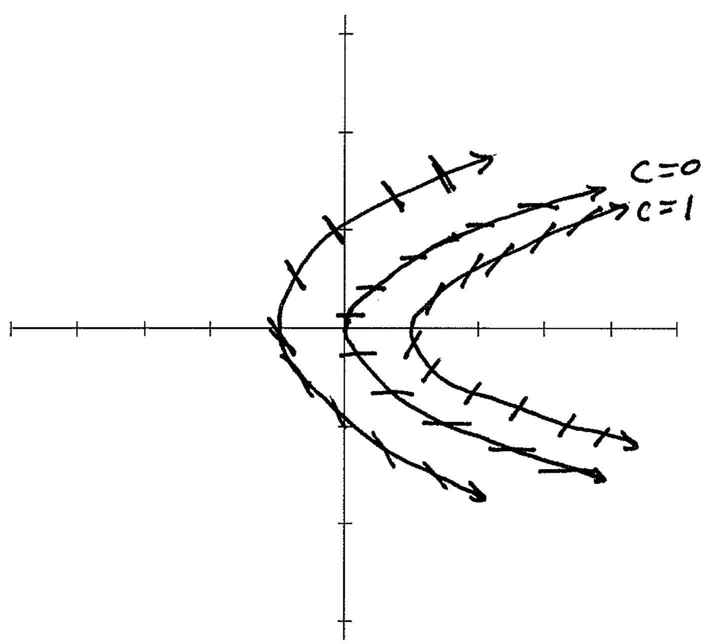
$$N(t) = P - Ce^{-kt}$$

11.

a) Sketch the solution curves on the following vector field that correspond to the initial conditions $y(0) = 1$ and $y(0) = -1$. How do the two solutions differ?



b) Describe the isoclines of the differential equation $\frac{dy}{dx} = x - y^2$. Draw a sketch showing three of the isoclines, each marked with short line segments having the appropriate slope.



$x - y^2 = c$

$y^2 = x - c$

$c = 0$

$y^2 = x$

$y = \pm\sqrt{x}$

$c = 1$

$y^2 = x - 1$

$y = \pm\sqrt{x-1}$

$c = -1$

$y^2 = x + 1$

$y = \pm\sqrt{x+1}$

12. The direction field drawn in problem 11 a) corresponds to the differential equation $y' = \cos x + y$. Find the general solution of this equation. Find particular solutions satisfying the initial conditions $y(0) = 1$ and $y(0) = -1$.

$y' - y = \cos x \qquad \rho(x) = e^{\int -1\, dx} = e^{-x}$

$e^{-x}y' - ye^{-x} = e^{-x}\cos x$

$\frac{d}{dx}(ye^{-x}) = e^{-x}\cos x$

$ye^{-x} = \int e^{-x}\cos x + C$

Do integration by parts twice!

$\int e^{-x}\cos x\, dx$
$= \frac{1}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x$

$ye^{-x} = \frac{1}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x + C$

$y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + Ce^{x}$

$y(0) = 1$

$1 = y(0) = \frac{1}{2}\cancel{\sin 0} - \frac{1}{2}\cos 0 + Ce^{0}$

$1 = -\frac{1}{2} + C$

$\Rightarrow C = 3/2$

$y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + \frac{3}{2}e^{x}$

$y(0) = -1$

$-1 = y(0) = \frac{1}{2}\cancel{\sin 0}^{0} - \frac{1}{2}\cos 0 + Ce^{0}$

$-1 = -\frac{1}{2} + C$

$C = -\frac{1}{2}$

$y = \frac{1}{2}\sin x - \frac{1}{2}\cos x - \frac{1}{2}e^{x}$