Cauchy-Euler Equations

A second-order differential equation of the form $t^2y'' + \alpha ty' + \beta y = 0$ is called a **Cauchy-Euler equation**. The exponent on the variable is the same as the degree of the derivative. To solve this equation, we will assume the solution has the form $y = t^m$. Under that assumption, we have

$$y' = mt^{m-1} \text{ and } y'' = m(m-1)t^{m-2}.$$

If we substitute this back into the original equation, we have

$$t^2m(m-1)t^{m-2} + \alpha tmt^{m-1} + \beta t^m = 0, \text{ or}$$

$$m(m-1)t^m + \alpha mt^m + \beta t^m = 0.$$

If $t^m \neq 0$, then we have the characteristic equation

$$m(m-1) + \alpha m + \beta = 0.$$

Derivation:

Let $y = u(x)$ and $x = \ln t$ so that $e^x = t$. Then, by the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = u'\frac{1}{t} \text{ or } u' = t\frac{dy}{dt}$$

Similarly,

$$\frac{d^2y}{dt^2} = \frac{d}{dx}\left(u'e^{-x}\right)\frac{dx}{dt} = \left(u''e^{-x} - u'e^{-x}\right)\frac{1}{t} = \left(u''\frac{1}{t} - u'\frac{1}{t}\right)\frac{1}{t} \text{ or } t^2\frac{d^2y}{dt^2} = u'' - u'.$$

With this substitution, the equation $t^2y'' + \alpha ty' + \beta y = 0$ becomes $u'' - u' + \alpha u' + \beta u = 0$ or $u'' + (\alpha - 1)u' + \beta u = 0$. This is a second order linear equation with constant coefficients.

Note: This substitution implies $x = \ln t$, so that $t = e^x$, and $t^m = e^{mx}$. This last equation is used to write the final solutions in a familiar form.