Exact Equations and Integrating Factors

Definition: A differential equation written in differential form $M(x,y)dx + N(x,y)dy = 0$ is **exact** if there is a differentiable function $Q(x,y)$ for which $\frac{\partial Q}{\partial x} = M(x,y)$ and $\frac{\partial Q}{\partial y} = N(x,y)$. If this is the case, the equation can be restated as $\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy = 0$ or $dQ = 0$ in which case $Q(x,y) = C$.

Theorem: Let M, N , $\frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ be continuous functions of x and y in a rectangle R of the xy-plane. Then the equation $M(x,y)dx + N(x,y)dy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 1: Identifying Exact Equations

Determine if the equation is exact.

a) $\left(2x + y^2\right)dx + \left(2xy + 1\right)dy = 0$

b) $\left(2x + y^2\right)dx + \left(2xy + 1\right)dy = 0$

c) $\frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy = 0$ Hint: Be careful at the origin.

Solving an Exact Equation

If $M(x,y)dx + N(x,y)dy = 0$ is exact, then there is a function $Q(x,y)$ for which $\frac{\partial Q}{\partial x} = M(x,y)$ and $\frac{\partial Q}{\partial y} = N(x,y)$.

- Integrate $Q(x,y) = \int M(x,y)dx = f(x,y) + h(y)$. The function $h(y)$ is used because the unknown constant might be a function of *y*.
- Use the fact that $\frac{\partial Q}{\partial y} = N(x,y)$ and take the partial derivative of $Q(x,y) = \int M(x,y)dx = f(x,y) + h(y)$ to get $\frac{\partial}{\partial y}\left(f(x,y) + h(y) \right) = N(x,y)$ or $\frac{\partial}{\partial y} f(x,y) + h'(y) = N(x,y)$.
- Solve the resulting differential equation for the unknown function $h(y)$.

Example 2: Solving an Exact Equation

Solve the exact equation $\left(2x + y^2\right)dx + \left(2xy + 1\right)dy = 0$.

Example 3: Solving an Exact Equation

Solve the exact equation $\left(y\cos x+y^2\right)dx+\left(\sin x+2xy+3y^2\right)dy=0$.

Example 4: Solving an Exact Equation

Show that the equation $\frac{y}{x}dx+\left(\ln x\right)dy=0$ is exact and solve.

Integrating Factors

We can make some equations exact by using an integrating factor. Suppose we have the equation $M(x,y)dx + N(x,y)dy = 0$ and this equation is not exact. We seek to find a function $\mu(x,y)$ such that $\mu M(x,y)dx + \mu N(x,y)dy = 0$ is an exact equation.

Example 5: An Integrating Factor

Show that the equation $ydx - xdy = 0$ becomes exact when multiplied through by $\phi(x,y) = \dfrac{1}{x^2+y^2}$.

To determine a method for finding an integrating factor. Start with $Mdx + Ndy = 0$ and multiply by μ to get $\mu Mdx + \mu Ndy = 0$. This equation must be exact so we must have $\dfrac{\partial}{\partial y}(\mu M) = \dfrac{\partial}{\partial x}(\mu N)$ so that $\mu\dfrac{\partial M}{\partial y} + M\dfrac{\partial \mu}{\partial y} = \mu\dfrac{\partial N}{\partial x} + N\dfrac{\partial \mu}{\partial x}$.

Suppose that μ is a function of x alone then we have $\mu\dfrac{\partial M}{\partial y} = \mu\dfrac{\partial N}{\partial x} + N\dfrac{d\mu}{dx}$ or $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)dx = \dfrac{d\mu}{\mu}$. If the left side of the equation is a function of x alone then the integrating factor is $\mu(x) = e^{\int f(x)dx}$ where $f(x) = \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$. A similar argument can be used to find an integrating factor if μ is a function of y only.

Rule: For the equation $Mdx + Ndy = 0$

- If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ is a function of x only, then $\mu(x) = e^{\int f(x)dx}$ is an integrating factor for the equation.
- If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$ is a function of y only, then $\mu(y) = e^{-\int g(y)dy}$ is an integrating factor for the equation.

Example 6: Solving an Equation using Integrating Factors

Solve $\frac{dy}{dx} = a(x)y + b(x)$ which is the same as $\left(a(x)y + b(x)\right)dx - dy = 0$

Example 7: Solving using an Integrating Factor

Solve $(4xy+3y^2-x)dx+x(x+2y)dy=0$ `