

First Order Linear Differential Equations

Definition: A **first order linear differential equation** is a differential equation that can be written in the form $\frac{dy}{dt} + f(t)y = g(t)$. $g(t)$ is the driving term (or input). The equation is said to be undriven (homogeneous) if $g(t) = 0$.

Example 1: First order linear differential equations

The following are examples of first order linear differential equations. Write each in the form given above and identify the functions $f(t)$ and $g(t)$.

a) $ty' + y = 2t$

b) $\frac{dy}{dt} + 2y = 3$

c) $\frac{dy}{dt} = y + \cos t$

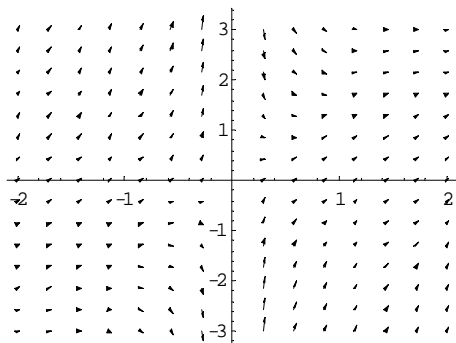
Example 2: Solving a first order linear differential equation

Solve $ty' + y = 2t$.

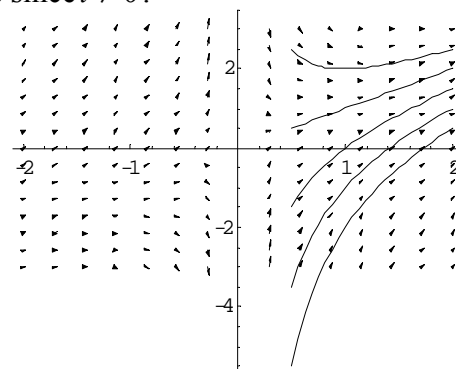
Solution: We notice that the left-hand side of the equation is the derivative of the product ty since $\frac{d(ty)}{dt} = ty' + y$. Using this we can rewrite the original equation as the derivative of a product and then integrate both sides: $(ty)' = 2t$

$$\begin{aligned}\int (ty)' dt &= \int 2t dt \\ (ty) &= t^2 + C \\ y &= t + \frac{C}{t}\end{aligned}$$

Note: This actually defines two solutions since $t \neq 0$.



Direction field for $ty' + y = 2t$



Direction field with some integral curves

Example 3: Solving a first order linear differential equation

Solve $\frac{dy}{dt} + 2y = 3$.

Solution: Try to reduce this problem to the type we solved in Example 2. Can we find a function $\mu(t)$ so that when we multiply both sides of the equation by $\mu(t)$ the left-hand side is the derivative of a product?

$$\mu(t) \left(\frac{dy}{dt} + 2y \right) = 3\mu(t)$$

$$\mu(t) \frac{dy}{dt} + \mu(t) 2y = 3\mu(t)$$

In order for the left-hand side to be the derivative of a product we must have $\mu'(t) = 2\mu(t)$. The only function for which this is true is $\mu(t) = e^{2t}$. With this choice of $\mu(t)$ the equation becomes

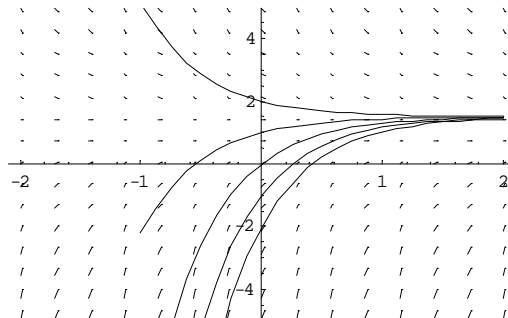
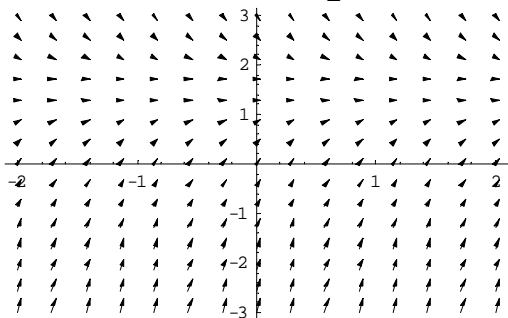
$$e^{2t} \frac{dy}{dt} + e^{2t} 2y = 3e^{2t}.$$

The left-hand side is the derivative of $e^{2t} y$. So we have

$$(e^{2t} y)' = 3e^{2t}$$

Integration of both sides yields $\int (e^{2t} y)' dt = \int 3e^{2t} dt$, so that $(e^{2t} y) = \frac{3}{2} e^{2t} + C$.

Solving for y : $y = \frac{3}{2} + Ce^{-2t}$



General Procedure for Solving First Order Linear Differential Equations

An equation of the form $\frac{dy}{dt} + p(t)y = g(t)$ can be solved by multiplying through by the function $\mu(t) = e^{\int p(t)dt}$ to get $e^{\int p(t)dt} \frac{dy}{dt} + e^{\int p(t)dt} p(t)y = g(t)e^{\int p(t)dt}$. The left-hand side of this equation is the derivative of a product: $\frac{d}{dt} \left(e^{\int p(t)dt} y \right) = g(t)e^{\int p(t)dt}$

Integrating both sides gives $e^{\int p(t)dt} y = \int g(t)e^{\int p(t)dt} dt$. Therefore, the solution is $y = e^{-\int p(t)dt} \int g(t)e^{\int p(t)dt} dt$. There is no need to memorize this formula, just learn the general procedure.

Steps in solving a first order linear equation using an integrating factor:

1. Put the equation in standard form $\frac{dy}{dt} + p(t)y = g(t)$
2. Compute the integrating factor $\mu(t) = e^{\int p(t)dt}$.
3. Solve the equation $(\mu y)' = \mu g$.

Example 4: Solving a first order linear differential equation

Solve $t \frac{dy}{dt} - y = t^3$.

Example 5: Solving a first order linear differential equation

Solve the initial value problem $\frac{dy}{dx} = y + \cos x$, $y(0) = 0$.

It is not always possible to integrate the right-hand side of $e^{\int p(t)dt} y = \int g(t)e^{\int p(t)dt} dt$ when solving. In the case of an initial value problem with $y(t_0) = y_0$, we can take a definite integral from t_0 to t .

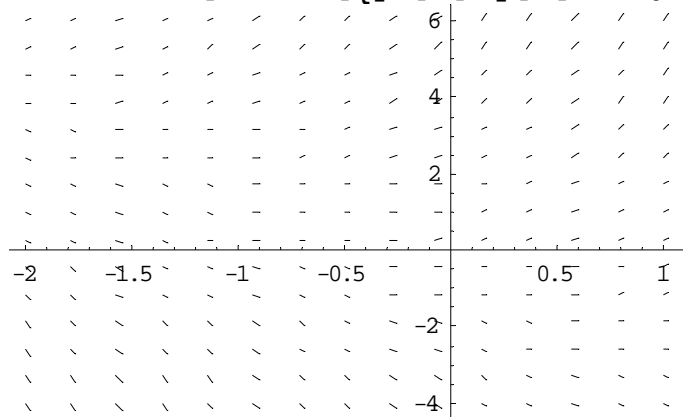
Example 6: Solving a first order linear differential equation

Solve the initial value problem $\frac{dy}{dt} + y = \frac{1}{1+t^2}$, $y(2) = 3$.

Mathematica Commands: Direction Field and Integral Curves for $y' = y + 2t$

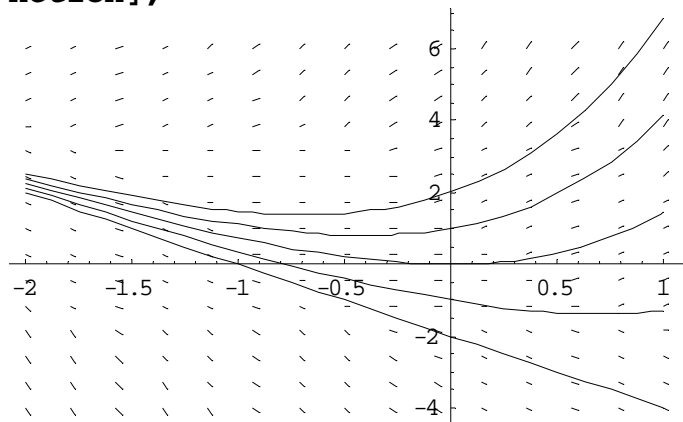
```
<<Graphics`PlotField`vf=PlotVectorField[{1,2t+y},{t,2,1},{y,4,6},Axes→Automatic,HeadLength→0,AspectRatio→1/GoldenRatio,DisplayFunction→$DisplayFunction];
```

```
sols=Table[DSolve[{y'[t]==y[t]+2*t, y[0]==k},y[t],t],{k,2,2}]
```



```
{{{y[t] → -2 (1 + t)}}},
{{y[t] → -2 + et - 2 t}}, {{y[t] → 2 (-1 + et - t)}}},
{{y[t] → -2 + 3 et - 2 t}}, {{y[t] → 2 (-1 + 2 et - t)}}}}
```

```
Do[g[k]=Plot[sols[[k,1,1,2]],{t,-2,1},PlotRange→All,DisplayFunction→Identity],{k,1,5} ]
Show[g[1],g[2],g[3],g[4],g[5],vf,DisplayFunction→$DisplayFunction];
```



Note: This Mathematica notebook is on the website.