Differential Equations

Definition: A relationship between a function and its derivatives is called a **differential equation**. The highest-order derivative that appears is called the **order** of the differential equation. The general form of an **nth-order differential equation** with independent variable *x* and unknown function $y(x)$ is $f(x, y, y', y'', \ldots, y^{(n)}) = 0$ where *f* is a real-valued function of $n+2$ variables.

Example 1:

a) The equation $\frac{dy}{dx} = ky$ is a first-order differential equation.

b) The equation $\frac{d^2y}{dx^2} + y = 0$ is a second-order differential equation

These differential equations are called **ordinary differential equations** because the unknown function depends on a single independent variable. If the unknown function depends upon more than one independent variable, then the equation is called a **partial differential equation.**

Example 2: The equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{k}\frac{\partial y}{\partial t}$ is a partial differential equation because the unknown function *y* depends on both *x* and *t*.

We will have three important goals in our study of differential equations:

1. To discover the differential equation that describes a physical situation (modeling).
2. Find the solution – exact or approximate - of the equation (solving).
3. Interpret the solution (interpreting).

Example 3: The equation $\frac{dP}{dt} = kP$ is a first-order differential equation that is used to model the time rate of change of population $P(t)$ with constant birth and death rates. This equation is used because under those conditions the rate of change is proportional to the population.

Definition: We say that a function y **satisfies** or is a **solution** of the differential equation on an interval *I* if the function has the indicated derivatives and makes the equation true for all values of the independent variable in the interval *I*.

One nice aspect of the study of differential equations is that one can always verify that a given function is a solution by taking derivatives and substituting into the original equation and verifying that an identity results.

Example 4: The function $P(t) = Ce^{kt}$ is a solution of the differential equation $\frac{dP}{dt} = kP$.

To see this we find the derivative of the function $P(t) = Ce^{kt}$ and substitute into the equation.

$$P'(t) = \frac{dP}{dt} = Cke^{kt} = k\left(Ce^{kt}\right) = kP.$$

Notice that each choice of the arbitrary constant *C* gives us a different solution. Thus the solution defines an infinite family of solutions.

Example 5: Verifying that a Given Function is a Solution of an ODE
Show that the equation $\frac{dy}{dt} = -0.04y + 32$ has for one solution the function $y(t) = 800 + 70e^{-0.04t}$.

Example 6: Verifying that a Given Function is a Solution of an ODE
Verify that the functions $y_1 = e^t \cos t$ and $y_2 = e^t \sin t$ are both solutions of the differential equation $y'' - 2y' + 2y = 0$.

Note: It follows that the sum $y_1 + y_2$ is also a solution of the differential equation. Why?

As we have seen, an equation may have more than one solution. If we wish to have a specific solution, then we must have an initial condition – a condition on the value of the solution of the equation at some point.

Definition: A first-order equation together with an initial condition is called an **initial value problem**. A solution of the differential equation that also satisfies the initial condition is a solution of the initial value problem. A general first-order initial value problem is denoted by $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$.

Exercise 1: Verify that $y(x)$ satisfies the given differential equation. Then find a value of *C* so that $y(x)$ satisfies the initial value problem.

a) $y' = 2y$; $y(t) = Ce^{2t}$; $y(0) = 3$

b) $y' = t - y$; $y(t) = Ce^{-t} + t - 1$; $y(0) = 10$

c) $e^{y}y' = 1$; $y(t) = \ln(t + C)$; $y(0) = 0$

In this class we will learn to solve many types of equations. There are, however, three basic types of first-order linear equations that we will solve. The simplest type of equation we can solve is of the form $\frac{dy}{dt} = f(t)$. In this type of equation, the dependent variable does not appear on the right-hand side of the equation. This equation can be solved directly by integrating both sides of the equation (if possible). An equation of the form $\frac{dy}{dt} = g(y)$ is called an autonomous differential equation since the independent variable does not appear in the right-hand side. The third type of equation is equations of the form $\frac{dy}{dt} = f(t, y)$.

Definitions: Consider the first-order differential equation of the form $\frac{dy}{dt} = f(t)$. We can solve this equation by integrating both sides.

$$\frac{dy}{dt} = f(t)$$

$$y = y(t) = \int f(t)dt + C \qquad (1.1)$$

Equation (1.1) defines the **general solution** of the differential equation $\frac{dy}{dt} = f(t)$. We can obtain a **particular solution** of the initial value problem $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$ by substituting $t = t_0$ and $y = y_0$ into the initial value problem and solving for *C*.

Example 7: Solving an Initial Value Problem

Solve the initial value problem $\frac{dy}{dt} = e^{-2t}$, $t > 0$, $y(0) = 5$.

Example 8: Solving an Initial Value Problem

Solve the initial value problem $\frac{dy}{dt} = t\sqrt{t^2 + 9}$, $y(-4) = 0$.

Exercise 2: Solving Initial Value Problems

a) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{x(x+1)}$, $x > 0$.

b) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{1+x^2}$.

c) Can you solve $\frac{dy}{dx} = e^{-x^2/2}$, $y(0) = 0$?

d) Can you solve $\frac{dy}{dx} + f(x)y = 0$?