Laplace Transforms Part 2 Discontinuity, Shift, and Impulse

In this section, we examine the solution of differential equations whose forcing functions are defined piecewise.

Definition: A function $f(t)$ has a jump discontinuity a $t = a$ if both $\lim_{h\to 0^+} f(a+h) = f(a^+)$ and $\lim_{h\to 0^+} f(a-h) = f(a^-)$ exist but are different.

In order to express functions with discontinuities for transform purposes, we will make use of the unit step function.

Definition: The Unit Step Function or **Heavidside Step Function** $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$.

In general, we write $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$ for a step function that changes at from 0 to 1 at the value $t = c$.

Example 1: **Using unit step functions**

Sketch the graph of $u_a(t) - u_b(t)$.

Example 2: **Using unit step functions**

Sketch the graph of $u_3(t)f(t-3)$ where $f(t)=\sin t$.

Example 3: **Using unit step functions**

Sketch the graph of $(t-1)u_1(t)-2(t-2)u_2(t)+(t-3)u_3(t)$.

Theorem: **2nd Shifting Theorem:** If $F(s)=\mathcal{L}\left\{f(t)\right\}$ exists for $s>a\geq 0$, and if c is a positive constant, then $\mathcal{L}\left\{u_c(t)f(t-c)\right\}=e^{-cs}\mathcal{L}\left\{f(t)\right\}=e^{-cs}F(s)$, $s>a$ and
If $f(t)=\mathcal{L}^{-1}\left\{F(s)\right\}$, then $u_c(t)f(t-c)=\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\}$

Example 4: **Finding Laplace transforms involving step functions**

Find the Laplace transform of $f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$.

Example 5: **Finding Laplace transforms involving step functions**

Find the Laplace transform of $f(t) = \begin{cases} 0 & t < \pi \\ t - \pi & \pi \leq t < 2\pi \\ 0 & t > 2\pi \end{cases}$.

Example 6: **Finding inverse Laplace transforms involving step functions**

Find the inverse Laplace transform of $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

Example 7: **Finding inverse Laplace transforms**

Find the inverse Laplace transform of $F(s) = \dfrac{3!}{(s-2)^4}$.

Example 8: **Finding inverse Laplace transforms involving step functions**

Find the inverse Laplace transform of $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$

Example 9: **Finding inverse Laplace transforms involving step functions**

Find the inverse Laplace transform of $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

Solving Differential Equations with Discontinuous Forcing Functions

Example 10: **Solving a differential equations with a discontinuous forcing function**

Solve $x'' + 4x = f(t)$ where $f(t) = \begin{cases} \cos 2t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$, $x(0) = x'(0) = 0$

Solution: First, write $f(t)$ in terms of unit step functions:

$f(t) = [1 - u_{2\pi}(t)]\cos 2t = \cos 2t - u_{2\pi}(t)\cos 2(t - 2\pi)$

The last part is true because of the periodicity of the cosine function.

Next, take the transform of both sides of the equation:

$$x'' + 4x = \cos 2t - u_{2\pi}(t)\cos 2(t - 2\pi)$$

$$s^2 X(s) + 4X(s) = \frac{s}{s^2 + 4} - e^{-2\pi s}\frac{s}{s^2 + 4}$$

So that,

$$X(s) = \frac{s}{(s^2 + 4)^2} - e^{-2\pi s}\frac{s}{(s^2 + 4)^2}.$$

Now, $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2 + 4)^2}\right\} = \frac{t}{4}\sin 2t$ so

$x(t) = \frac{t}{4}\sin 2t - u_{2\pi}(t)\frac{1}{4}(t - 2\pi)\sin 2(t - 2\pi)$, or

$x(t) = \frac{1}{4}\sin 2t[t - u_{2\pi}(t)(t - 2\pi)]$

This can be written in the form: $x(t) = \begin{cases} \frac{1}{4}t\sin 2t & \text{if } t < 2\pi, \\ \frac{1}{2}\pi\sin 2t & \text{if } t \ge 2\pi \end{cases}.$

Solving Differential Equations with Discontinuous Forcing Functions

Example 11: **Solving a differential equations with a discontinuous forcing function**

Solve $\frac{d^2y}{dt^2} + 4y = 3u_5(t)\sin(t-5)$ where, $y(0) = 1, y'(0) = 0$.

Solution:

Take the transform of both sides: $\mathcal{L}\left[\frac{d^2y}{dt^2} + 4y\right] = \mathcal{L}\left[3u_5(t)\sin(t-5)\right]$, then using the linearity property, we have: $\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 4\mathcal{L}[y] = \mathcal{L}\left[3u_5(t)\sin(t-5)\right]$.

Thus

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{3e^{-5s}}{s^2+1}.$$

Substituting the initial conditions and simplifying, we have

$$\left(s^2+4\right)Y(s) - s = \frac{3e^{-5s}}{s^2+1}.$$

So

$$Y(s) = \frac{s}{s^2+4} + \frac{3e^{-5s}}{\left(s^2+1\right)\left(s^2+4\right)}.$$

Using partial fractions, we can write the equation as

$$Y(s) = \frac{s}{s^2+4} + \frac{e^{-5s}}{s^2+1} - \frac{e^{-5s}}{s^2+4}.$$

Then, taking inverse transforms, we have

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+4}\right\}.$$

Therefore, $y(t) = \cos 2t + u_5(t)\left[\sin(t-5) - \frac{1}{2}\sin(2(t-5))\right]$.