The Laplace Transform

Definition: An **integral transform** is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t) dt$$

The function F is said to be the **transform** of f and the function K is called the **kernel** of the transformation.

Definition: Let $f(t)$ be given for $t \geq 0$ and suppose that f satisfies certain conditions to be stated later. Then the **Laplace transform** of f, denoted by $\mathcal{L}\{f(t)\}$ is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t) dt .$$

This is an improper integral. If the limit defining the improper integral exists then we say the integral converges, otherwise, the integral is said to diverge.

Facts: The Laplace transform changes differentiation to multiplication and integration to division. This will be useful for solving differential equations.

Example 1: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = 1$.

$$\mathcal{L}\{f(t) = 1\} = F(s) = \int_{0}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{0}^{\infty} = \frac{1}{s}, \text{ for } s > 0 .$$

$$F(s) = \frac{1}{s}$$

Example 2: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = e^{at}$

$$F(s) = \frac{1}{s-a} \text{ for } s > a$$

Theorem: Existence of Laplace Transform

Let $f(t)$ be given for $0 < t < \infty$ and suppose $\lim_{t\to\infty} e^{-\alpha t} f(t) = 0$ for some α, then $F(s)$ exists for some values of *s*.

Properties of the Laplace Transform

1. The Laplace transform is linear. $\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\}$
2. If $f = u + iv$ then $\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\{u + iv\} = \mathcal{L}\{u\} + i\mathcal{L}\{v\}$
3. $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2 + \omega^2}$
4. $\mathcal{L}\{\sin \omega t\} = \dfrac{w}{s^2 + \omega^2}$
5. $\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0^+)$ where $f(0^+) = \lim_{h\to 0^+} f(h)$
6. $\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - sf(0^+) - f'(0^+)$
7. $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - \cdots - f^{(n-1)}(0^+)$
8. $\mathcal{L}\left\{\int_0^t f(x)dx\right\} = \dfrac{1}{s} F(s)$
9. $\mathcal{L}\left\{e^{bt} f(t)\right\} = F(s - b)$ First Shifting Theorem

Proof:

1. $\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt$

$$= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt = c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\}$$

2. Think about it!
3. From Example 2 with $f(t) = e^{i\omega t}$ we have $\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{e^{i\omega t}\right\} = \dfrac{1}{s - i\omega}$.

   Rationalizing the denominator and separating real and imaginary parts, we have

$$\mathcal{L}\{\cos \omega t + i \sin \omega t\} = \mathcal{L}\left\{e^{i\omega t}\right\} = \frac{1}{(s - i\omega)} \frac{(s + i\omega)}{(s + i\omega)} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}$$

   This shows that $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2 + \omega^2}$ and $\mathcal{L}\{\sin \omega t\} = \dfrac{w}{s^2 + \omega^2}$.

4. see #3.

5. By definition, $\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)dt$. Integrating by parts, we have

   $\int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt = -f(0^+) + sF(s)$ where

   $f(0^+) = \lim_{h\to 0^+} f(h)$

6. $\mathcal{L}\left\{f''(t)\right\} = s\mathcal{L}\left\{f'(t)\right\} - f'(0^+) = s\left(sF(s) - f(0^+)\right) - f'(0^+) = s^2F(s) - sf(0^+) - f'(0^+)$

7. Repeat the procedure used in #6.

8. Let $g(t) = \int_0^t f(x)dx$, then $g'(t) = f(t)$. It follows that

   $F(s) = \mathcal{L}\left\{g'(t)\right\} = s\mathcal{L}\left\{g(t)\right\} - g(0^+) = s\mathcal{L}\left\{\int_0^t f(x)dx\right\}$. Therefore,

   $\mathcal{L}\left\{\int_0^t f(x)dx\right\} = \frac{F(s)}{s}$

9. $\mathcal{L}\left\{e^{bt} f(t)\right\} = \int_0^\infty e^{bt}e^{-st} f(t)dt = \int_0^\infty e^{-(s-b)t} f(t)dt = F(s-b)$

Example 3: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = e^{3t}\cos 2t$

$\mathcal{L}\left\{\cos 2t\right\} = \frac{s}{s^2+4}$ so $\mathcal{L}\left\{e^{3t}\cos 2t\right\} = \frac{s-3}{(s-3)^2+4}$

Example 4: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = \sinh 3t$

(Hint: Recall that $\sinh t = \dfrac{e^{t} - e^{-t}}{2}$)

Solution: $\mathcal{L}\{\sinh 3t\} = \mathcal{L}\left\{\dfrac{e^{3t} - e^{-3t}}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\left\{e^{3t}\right\} - \frac{1}{2}\mathcal{L}\left\{e^{-3t}\right\} = \frac{1}{2}\frac{1}{(s-3)} - \frac{1}{2}\frac{1}{(s+3)} = \frac{3}{s^2 - 9}$$

Example 5: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = \cosh 3t$

Solution: We use the fact that if $f(t) = \cosh 3t$ then $f'(t) = 3\sinh 3t$.
Using the fact that $\mathcal{L}\left\{f'(t)\right\} = 3\mathcal{L}\{\sinh 3t\} = 3\dfrac{3}{s^2 - 9} = \dfrac{9}{s^2 - 9}$ and rule #5 above, we have

$$\mathcal{L}\left\{f'(t)\right\} = \frac{9}{s^2 - 9} = sF(s) - f(0^+) = s\mathcal{L}\{\cosh 3t\} - 1$$

So that, $\dfrac{9}{s^2 - 9} = s\mathcal{L}\{\cosh 3t\} - 1$.

Solving for $\mathcal{L}\{\cosh 3t\}$, we have $\mathcal{L}\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$

Example 6: **Finding a Laplace Transform**

Find the Laplace transform of $f(t) = t^2 e^{4t}$

Solution: Multiplying by e^{4t} shifts the transform of t^2 so that $\mathcal{L}\left\{t^2 e^{4t}\right\} = \mathcal{L}\left(t^2\right)(s-4)$.
The transform of t^2 is found as follows:
If we let $f(t) = t^2$, so that $f'(t) = 2t$ and $f''(t) = 2$, then example 1 shows that
$\mathcal{L}\left\{f''(t)\right\} = \mathcal{L}\{2\} = \dfrac{2}{s}$. Then by rule #6 , we have
$\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - sf(0^+) - f'(0^+) = s^2\mathcal{L}\left\{t^2\right\} - sf(0^+) - f'(0^+) = s^2\mathcal{L}\left\{t^2\right\}$.
Therefore, $s^2\mathcal{L}\left\{t^2\right\} = \dfrac{2}{s}$ or $\mathcal{L}\left\{t^2\right\} = \dfrac{2}{s^3}$.

This result can be generalized. If we let $f(t) = t^n$, then $\mathcal{L}\left\{f^{(n)}(t)\right\} = \dfrac{n!}{s}$. Combining this with rule #7, we would see that $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$

Applications to Initial Value Problems

Definition: The function $u(t) = \mathcal{L}^{-1}\left\{U(s)\right\}$ is called the **inverse Laplace transform**.

Theorem: The inverse Laplace transform is linear. That is,

$$\mathcal{L}^{-1}\left\{c_1F_1(s) + c_2F_2(s)\right\} = c_1\mathcal{L}^{-1}\left\{F_1(s)\right\} + c_2\mathcal{L}^{-1}\left\{F_2(s)\right\}.$$

Steps to Solving an Initial Value Problem Using The Laplace Transform

1. Transform the equation.
2. Solve for $U(s)$.
3. Find the corresponding $u(t) = \mathcal{L}^{-1}\left\{U(s)\right\}$.
   a) Look up $U(s)$ in tables.
   b) Express $U(s)$ as $V(s+b)$ and apply the shifting theorem.
   c) Decompose $U(s)$ by partial fractions.

Example 7: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\dfrac{dy}{dt} + 2y = 0$, $y(0) = 3$ using Laplace Transforms.

Example 8: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{dy}{dt} + 3y = 1$, $y(0) = -1$ using Laplace Transforms.

Example 9: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{d^2y}{dt^2} + y = 1$, $y(0) = 2$, $y'(0) = 0$ using Laplace Transforms.

Example 10: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 3$ using Laplace Transforms.

Example 11: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{dy}{dt} + 2y = e^{-2t}$, $y(0) = 0$ using Laplace Transforms.

Example 12: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 0$ using Laplace Transforms.

Example 13: **Solving an Initial Value Problem Using the Laplace Transform**

Solve the IVP $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$ using Laplace Transforms.