Laplace Transforms Part 3 Impulse

In this section, we examine the solution of differential equations whose forcing functions are impulse functions.

Definition: The Dirac-δ function $\delta(t)$ is defined to have the following properties

$\delta(t) = 0$ for all $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

The function $\delta(t - t_0)$ represents an impulse that is centered on time $t = t_0$ and can be defined as the limit of the function $D(t)$ as $T \to \infty$ where

$$D(t) = \begin{cases} 0 & t \leq t_0 - \frac{1}{2}T \\ \frac{1}{2}T & t_0 - \frac{1}{2}T < t < t_0 + \frac{1}{2}T \\ 0 & t \geq t_0 + \frac{1}{2}T \end{cases}.$$

Then, using the definition above we could deduce that the Dirac-δ function has the following filtering property

Definition: Filtering Property of Dirac-δ function $\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$ for any function $f(t)$ continuous on $\mathbb{R}$.

The **Laplace Transform of the Dirac-δ function** is $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$ in particular, $\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} \mathcal{L}\{\delta(t-t_0)\} = 1$.

Example 1: **Solving an IVP with an impulse forcing function**

Solve the initial value problem $\frac{d^2y}{dt^2} + 4y = 1 + \delta(t-1)$, $y(0) = 1 = y'(0)$ using Laplace transforms.

Example 2: **Solving an IVP with an impulse forcing function**

Solve the initial value problem $2\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=\delta(t-5)$, $y(0)=0, y'(0)=0$ using Laplace transforms.

Example 3: **Solving an IVP with an impulse forcing function**

Solve the initial value problem $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \delta(t-5) + u_{10}(t)$, $y(0) = 0, y'(0) = 1/2$ using Laplace transforms.