Mathematical Models

Unrestricted Growth Model

Rate of change is proportional to current population: $\frac{dP}{dt} = kP$

Radioactive Decay

Rate of change is proportional to amount present: $\frac{dQ}{dt} = -rQ$

Limited Growth Model
Rate of change is proportional to the difference between some maximum population M and the current population: $\frac{dP}{dt} = k(M - P)$

Logistic Growth Model
Rate of change is jointly proportional to the current population and the difference between some maximum population M and the current population: $\frac{dP}{dt} = kP(M - P)$
This models the spread of infectious diseases, the growth of a business, and the spread of a rumor.

Newton's Law of Cooling*:* The rate at which an object cools is proportional to the difference in temperature between the object and the surrounding environment: $\frac{dT}{dt} = k(T - T_a)$. We use *T* to represent the temperature of the object at any time *t* and T_a for the ambient temperature.

Radioactive Decay

If a radioactive substance A is not only decaying, but is also being formed by the decay of some other radioactive substance B, then the number N of nuclei of substance A present at time *t* is given by the differential equation: $\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t}$. Where λ and μ are the decay constants of A and B, respectively, and B_0 is the number of nuclei of B present at time $t = 0$. Note: We are making the assumption that there is only one decay route for the B nuclei so that the $|\text{rate of decrease of B}| = |\text{rate of increase of A}|$.

Electrical Circuit
An electrical circuit contains an inductor of inductance L, a resistor of resistance R, and an ideal voltage generator that produces an alternating voltage $E\cos(\omega t)$. The differential equation determining the electrical current in the circuit at time t is given by $L\frac{dI}{dt} + RI = E\cos(\omega t)$

Growth Model: Constant Birth and Death rates

Rate of change is proportional to difference between birth rate α and death rate β :

$$\frac{dP}{dt} = \alpha P - \beta P = (\alpha - \beta) P$$

Growth Model: Limited Food Supply

Rate of change is proportional to difference between birth rate α and death rate β but assume death rate is directly proportional to population $\beta = \beta_1 P$ (more people die as population increases because of lack of food) $\frac{dP}{dt} = \alpha P - \beta P = (\alpha - \beta_1 P)P$

Growth Model: Population Explosion (also a model for nuclear fission)

Rate of change is proportional to difference between birth rate α and death rate β : but assume birth rate is directly proportional to population $\alpha = \alpha_1 P$ (more people die as population increases because of lack of food) $\frac{dP}{dt} = \alpha P - \beta P = (\alpha_1 P - \beta) P$

Unrestricted Growth with Constant Harvesting

Rate of change is proportional to current population with harvesting at a constant rate *H*:

$$\frac{dP}{dt} = kP - H$$

Let $P(t)$ represent the population of a species introduced to a new area for harvesting purposes. Suppose we introduce $P(0) = 100$ individuals and suppose the population grows exponentially with growth-rate coefficient $k = 2$ if there is no harvesting. How should we set the harvesting rate C in the model so that the population remains steady at $P(t) = 100$ individuals? Can we use the same rate forever or will it have to be adjusted?

1. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

2. The time rate of change of an alligator population $P(t)$ in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?