Practice Exam 2 Differential Equations

1. Solve the initial value problem: $4y'' - 8y' + 3y = 0$, $y(0) = 2, y'(0) = \frac{1}{2}$
2. Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of the equation: $2t^2y'' + 3ty' - y = 0$, $t > 0$.
3. Find the general solution of $y'' + y' + y = 0$.
4. Find the general solution of $y'' - 3y' - 4y = 2\sin t$
5. Solve the initial value problem $y^{(4)} - y = 0$,
   $y(0) = \frac{7}{2}, y'(0) = -4, y''(0) = \frac{5}{2,} y'''(0) = -2$.
6. Solve the Euler- Cauchy equation $\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0$ by assuming the solution has the form $y = t^m$ and substituting into the differential equation.
7. Find a particular solution of $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \cos x$
8. Use variation of parameters to find a particular solution of $2\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \sqrt{t}$.
9. Find a *particular solution* of $y'' - y = \frac{1}{e^t + e^{-t}}$, using reduction of order and the fact that $y(t) = e^t$ is a solution of the homogeneous equation..
10. Solve the system of linear differential equations $\begin{aligned}\frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y\end{aligned}$ in two different ways:
    a) By using elimination
    b) By using eigenvalues and eigenvectors
11. Solve the system of linear differential equations $\begin{aligned}\frac{dx}{dt} &= -x \\ \frac{dy}{dt} &= x - y\end{aligned}$ in two different ways:
    a) By using elimination
    b) By using eigenvalues and eigenvectors

12. Solve the initial value problem: $y'' + 4y' + 5y = 8\sin x$, $y(0) = 0, y'(0) = 0$
13. The charge *q* on the capacitor in an RCL circuit is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$.
    Supposes a given circuit contains a resistance of 10 ohms, an inductance of 10^{-4} henry, and a capacitance of 10^{-6} farad. Find the steady state complex charge if the impressed voltage is $v_0 \cos \omega t$. (Hint: Instead of using the cosine function, use the complex voltage $V(t) = v_0 e^{j\omega t}$, where $j = \sqrt{-1}$)

Solutions

1. $y(t) = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}$

2. Show they satisfy the equation then show that $W(y_1, y_2) = -\frac{3}{2}t^{-3/2} \neq 0$ if $t > 0$.

3. $y(t) = c_1 e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + c_2 e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)$

4. $y\,(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t$

5. $y(t) = 3e^{-t} + \frac{1}{2}\cos t - \sin t$

6. $y(t) = c_1 t^2 + c_2 t$

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8. $y_p(t) = \left[\int_0^t -2\sqrt{u}e^{u/2}\sin\left(\frac{u}{2}\right)\right]\left[e^{-t/2}\cos\left(\frac{t}{2}\right)\right] + \left[\int_0^t 2\sqrt{u}e^{u/2}\cos\left(\frac{u}{2}\right)\right]\left[e^{-t/2}\sin\left(\frac{t}{2}\right)\right]$

9. $y_p(t) = -\frac{t}{2}e^{-t} - \frac{1}{4}\left(e^t + e^{-t}\right)\ln(1 + e^{-2t})$

10. $$\begin{aligned} x(t) &= c_1 e^{-t} + 3c_2 e^{4t} \\ y(t) &= -c_1 e^{-t} + 2c_2 e^{4t} \end{aligned}$$

11. $$\begin{aligned} x(t) &= c_2 e^{-t} \\ y(t) &= c_1 e^{-t} + c_2 t e^{-t} \end{aligned}$$ or $\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} t e^{-t}$

12. $y(x) = e^{-2x}\cos x + e^{-2x}\sin x + \sin x - \cos x$

13. $q_c(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ where $\lambda_{1,2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$. It follows that if $R \neq 0$ then $q_c(t) \to 0$ as $t \to \infty$. Try $q_p(t) = Ae^{j\omega t}$, then

$$q_p(t) = \frac{v_0}{\left(\frac{1}{c} - L\omega^2\right) + jR\omega}e^{j\omega t}$$ and the steady-state complex charge is

$$q(t) = \frac{v_0}{\left(10^6 - 10^{-4}\omega^2\right) + 10j\omega}e^{j\omega t}$$

Solutions:

1. Solve the initial value problem: $4y'' - 8y' + 3y = 0$, $y(0) = 2, y'(0) = \frac{1}{2}$

   Solution: The equation is homogeneous and has characteristic equation $4\lambda^2 - 8\lambda + 3 = 0$ or $(2\lambda - 3)(2\lambda - 1) = 0$. This has solutions two distinct solutions $\lambda_{1,2} = \frac{3}{2}, \frac{1}{2}$. The general solution has the form $y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$. The initial condition that $y(0) = 2$ implies that $c_1 + c_2 = 2$. Now, $y'(t) = \frac{3}{2} c_1 e^{3t/2} + \frac{1}{2} c_2 e^{t/2}$ so the condition that $y'(0) = \frac{1}{2}$ implies that $\frac{3}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2}$ or $3c_1 + c_2 = 1$. Solving this system of equations for c_1 and c_2 gives $c_1 = -\frac{1}{2}$ and $c_2 = \frac{5}{2}$. Then, the particular solution is $y(t) = -\frac{1}{2} e^{3t/2} + \frac{5}{2} e^{t/2}$

2. See note above.
3. Find the general solution of $y'' + y' + y = 0$.

   Solution: The equation has characteristic equation $\lambda^2 + \lambda + 1 = 0$. This has complex conjugate solutions $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. The general solution has the form $y(t) = c_1 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)$.
4. Find the general solution of $y'' - 3y' - 4y = 2\sin t$

   Solution: First, solve the homogeneous equation which has characteristic equation $\lambda^2 - 3\lambda - 4 = 0$. This equation has solutions $\lambda_{1,2} = -1, 4$, so the complementary solution is $y_c(t) = c_1 e^{4t} + c_2 e^{-t}$. Since the forcing function is $f(t) = 2\sin t$ we guess that the particular solution is of the form $y_p(t) = A\cos t + B\sin t$. This function has derivatives $y_p'(t) = -A\sin t + B\cos t$ and $y_p''(t) = -A\cos t - B\sin t$. Substituting into the original equation, we have

$$\begin{gathered} y'' - 3y' - 4y = 2\sin t \\ -A\cos t - B\sin t - 3(-A\sin t + B\cos t) - 4(A\cos t + B\sin t) = 2\sin t \\ -A\cos t - B\sin t + 3A\sin t - 3B\cos t - 4A\cos t - 4B\sin t = 2\sin t \\ (-5A - 3B)\cos t + (-5B + 3A)\sin t = 2\sin t \end{gathered}$$

   This results in the system of equations: $\begin{matrix} -5A - 3B = 0 \\ 3A - 5B = 2 \end{matrix}$ for the coefficients. This has solutions $A = \frac{3}{17}$ and $B = \frac{-5}{17}$. The general solution is

$$y(t) = y_c(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t$$

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7. **Solution**: This is a Cauchy-Euler equation. If we let $y = x^m$ and substitute into the original equation, we find that $m = \pm 1$. So the complementary solution is of the form $y_c(x) = c_1 x + c_2 x^{-1}$. To find the second solution using variation of parameters , we seek a solution of the form $y_p(t) = v_1 y_1 + v_2 y_2$ where v_1, v_2 are found using

$v_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}}{W}$ and $v_2' = \dfrac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}}{W}$. The two functions $y_1 = x$ and $y_2 = x^{-1}$ have Wronskian $W(y_1, y_2) = -\dfrac{2}{x}$. Then, $v_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}}{W} = \dfrac{\begin{vmatrix} 0 & x^{-1} \\ \cos x & -x^{-2} \end{vmatrix}}{-2/x} = \dfrac{1}{2}\cos x$ and

$v_2' = \dfrac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}}{W} = \dfrac{\begin{vmatrix} x & 0 \\ 1 & \cos x \end{vmatrix}}{-2/x} = -\dfrac{x^2}{2}\cos x$. Integrating, we find that $v_1 = \dfrac{1}{2}\sin x$ and $v_2 = \left(1 - \dfrac{x^2}{2}\right)\sin x - x\cos x$. The particular solution is therefore

$$y_p(t) = v_1 y_1 + v_2 y_2 = \frac{1}{2} x \sin x + \left[\left(1 - \frac{x^2}{2}\right)\sin x - x\cos x\right] x^{-1}$$
$$= x^{-1}\sin x - \cos x .$$

The general solution is $y(x) = y_c(x) + y_p(x) = c_1 x + c_2 x^{-1} + x^{-1}\sin x - \cos x$.

8. Use variation of parameters to find a particular solution of $2\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = \sqrt{t}$.

Solution: Begin by finding the solutions of the homogeneous (complementary) equation: $2\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$. This has the characteristic equation $2\lambda^2 + 2\lambda + 1 = 0$ or $\lambda^2 + \lambda + \dfrac{1}{2} = 0$. This has the two complex conjugate solutions $\lambda_{1,2} = -\dfrac{1}{2} \pm \dfrac{1}{2} i$. The two solutions are $y_1(t) = e^{-\frac{1}{2}t}\cos\left(\dfrac{t}{2}\right)$ and $y_2(t) = e^{-\frac{1}{2}t}\sin\left(\dfrac{t}{2}\right)$. Their Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) & e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) \\ -\frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) - \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) & \frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) - \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) \end{vmatrix}$$

$$= \frac{1}{2}e^{-t}\cos^2\left(\frac{t}{2}\right) - \frac{1}{2}e^{-t}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right) + \frac{1}{2}e^{-t}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right) + \frac{1}{2}e^{-t}\sin^2\left(\frac{t}{2}\right)$$

$$= \frac{1}{2}e^{-t}\cos^2\left(\frac{t}{2}\right) + \frac{1}{2}e^{-t}\sin^2\left(\frac{t}{2}\right) = \frac{1}{2}e^{-t}\left(\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)\right) = \frac{1}{2}e^{-t}.$$

The solution we seek is of the form $y_p(t) = v_1 y_1 + v_2 y_2$ where v_1, v_2 are found using $v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}}{W}$ and $v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}}{W}$. In this case, these are given by

$$v_1' = \frac{\begin{vmatrix} 0 & e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) \\ \sqrt{t} & \frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) - \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) \end{vmatrix}}{\frac{1}{2}e^{-t}} = -2e^t\left(\sqrt{t}e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right)\right) = -2\sqrt{t}e^{\frac{1}{2}t}\sin\left(\frac{t}{2}\right), \text{ and}$$

$$v_2' = \frac{\begin{vmatrix} e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) & 0 \\ -\frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) - \frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\frac{t}{2}\right) & \sqrt{t} \end{vmatrix}}{\frac{1}{2}e^{-t}} = 2e^t\sqrt{t}e^{-\frac{1}{2}t}\cos\left(\frac{t}{2}\right) = 2\sqrt{t}e^{\frac{1}{2}t}\cos\left(\frac{t}{2}\right)$$

Now write the answer in terms of integrals.

11. Solve the system of linear differential equations $\begin{aligned}\frac{dx}{dt} &= -x \\ \frac{dy}{dt} &= x - y\end{aligned}$ in two different ways:

Solution:
Using elimination:
Solve the second equation for *x*: $x = y' + y$ then take the derivative to get $x' = y'' + y'$. Substitute both of these into the first equation to get

$$\frac{dx}{dt} = -x$$
$$y'' + y' = -y' - y$$

$$y'' + 2y' + y = 0.$$

This last equation has characteristic equation $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$. This has one (double) root $\lambda = -1$. So $y(t) = c_1 e^{-t} + c_2 t e^{-t}$. In order to find $x(t)$, we take the derivative of y, $y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$, and use the fact that $x = y' + y$:

$x = y' + y = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + c_1e^{-t} + c_2te^{-t} = c_2e^{-t}$. So the solution is:

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2e^{-t} \\ c_1e^{-t} + c_2te^{-t} \end{bmatrix}$$

Using eigenvalues and eigenvectors:

The associated matrix is $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$. It follows that $(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}$.

This has characteristic equation $(-1-\lambda)^2 = 0$, so there is a single eigenvalues $\lambda = -1$. The eigenvector corresponding to this is given by the matrix equation $(\mathbf{A} + \mathbf{I})\,\mathbf{v}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It follows that $v_1 = 0$ and we can choose $v_2 = 1$.

The eigenvector corresponding to $\lambda = -1$ is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The first solution is $\mathbf{x}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t}$

Next, we seek a second solution of the form $\mathbf{x}_2(t) = (\mathbf{v}_1t + \mathbf{v}_2)e^{-t}$. To find $\mathbf{v}_2$ we solve the equation $(\mathbf{A} + \mathbf{I})\,\mathbf{v}_2 = \mathbf{v}_1$: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This yields $v_1 = 1$ and we choose $v_2 = 0$.

With these choices, the second solution has the form

$$\mathbf{x}_2(t) = (\mathbf{v}_1t + \mathbf{v}_2)e^{-t} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{-t}.$$

The general solution is then

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t} + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{-t} = \begin{bmatrix} c_2e^{-t} \\ c_1e^{-t} + c_2te^{-t} \end{bmatrix}$$