Reduction of Order

When solving an equation of the form $a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0$ sometimes the characteristic equation will have only one root. This will give us a single solution of the form $e^{\lambda t}$. We must find a second solution. The method of reduction of order developed by D'Alembert allows us to find the other needed solution.

We assume the solution has the form $y(t)=v(t)e^{\lambda t}$, substitute into the original equation, and solve for $v(t)$

Example 1: Repeated Roots and Reduction of Order

Solve $y''+4y'+4y=0$.

Solution: The characteristic equation is $\lambda^2+4\lambda+4=0$. This equation has the double root $\lambda=-2$. This gives us one solution $y_1(t)=e^{-2t}$. The method of reduction of order requires us to make a guess for the second solution in the form $y_2(t)=v(t)e^{-2t}$.

Taking derivatives, we have $y_2{}'(t)=v'(t)\left(e^{-2t}\right)-2v(t)\left(e^{-2t}\right)$,
and

$$y_2{}''(t)=v''(t)\left(e^{-2t}\right)-4v'(t)\left(e^{-2t}\right)+4v(t)\left(e^{-2t}\right)$$

Substituting these into the original equation, we have

$$y''+4y'+4y=0$$

$$v''(t)\left(e^{-2t}\right)-4v'(t)\left(e^{-2t}\right)+4v(t)\left(e^{-2t}\right)+4\left[v'(t)\left(e^{-2t}\right)-v(t)\left(2e^{-2t}\right)\right]+4\left[v(t)e^{-2t}\right]=0$$

$$v''(t)\left(e^{-2t}\right)-4v'(t)\left(e^{-2t}\right)+4v(t)\left(e^{-2t}\right)+4v'(t)\left(e^{-2t}\right)-8v(t)\left(e^{-2t}\right)+4v(t)\left(e^{-2t}\right)=0$$

$$v''(t)-4v'(t)+4v(t)+4v'(t)-8v(t)+4v(t)=0$$

$$v''(t)=0$$

In this case, $v''(t)=0\Rightarrow v'(t)=c_1$, and $v'(t)=c_1\Rightarrow v(t)=c_1t+c_2$.

Therefore, the solution has the form $y(t)=c_1te^{-2t}+c_2e^{-2t}$. A check of the Wronskian shows that these solutions are linearly independent.

Reduction of Order can also be used to find a second solution of an equation if one is known.

Example 2: Using Reduction of Order

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Solution: We assume the solution has the form $y(t) = vt^{-1}$. Next, we take first and second derivatives:

$$y' = v\left(-t^{-2}\right) + v't^{-1} = v't^{-1} - vt^{-2}$$

$$y'' = v'\left(-t^{-2}\right) + v\left(2t^{-3}\right) - v't^{-2} + v''t^{-1}$$

$$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

Next, we substitute these into the original equation $t^2y'' + 3ty' + y = 0$.

$$t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) + vt^{-1} = 0$$

$$tv'' - 2v' + 2t^{-1}v + 3v' - 3vt^{-1} + vt^{-1} = 0$$

$$tv'' + v' = 0$$

If we make the change of variables $w = v'$, so that $w' = v''$, then and the equation becomes

$$w' + \frac{1}{t}w = 0$$

This separable equation has solution $w = \frac{1}{t}$ or $v' = \frac{1}{t}$.Then,

$$v(t) = \int \frac{1}{t}dt = \ln t\,.$$

Therefore, $y(t) = vt^{-1} = t^{-1}\ln t$.