Matrices

Definition: A **matrix** is a rectangular array of numbers. We designate the name of the matrix with a capital letter such as A.

Example 1: A matrix A is given by $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Definitions: The numbers in a given matrix are called the **elements** of the matrix. The elements are enclosed in brackets and arranged into (horizontal) rows and (vertical) columns. Each matrix has a dimension associated with it. The **dimension** of any matrix A is denoted $m \times n$ where m is the number of rows and n is the number of columns of A. The elements in a matrix can be identified with double-subscript notation; a_{ij} designates the element in the matrix in the ith row and jth column.

Example 2:

1. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has 2 rows and 2 columns and is a 2×2 matrix with elements $a_{11} = 1$, $a_{12} = 2$, $a_{21} = 3$, $a_{22} = 4$.
2. The matrix $B = \begin{bmatrix} -2 & 3 & 9 \\ -1 & 2 & -2 \\ 1 & 2 & 3 \end{bmatrix}$ is a 3×3 matrix. Some of the elements of B include $b_{11} = -2, b_{12} = 3, b_{13} = 9, b_{22} = 2, b_{31} = 1$.

   Each of the matrices in (1) and (2) above are called **square matrices**.

3. The matrix $E = \begin{bmatrix} a & b & c & d \end{bmatrix}$ is a 1×4 **row vector.**
4. The matrix $F = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$ is a 3×1 **column vector**.

Basic Matrix Operations

Definition: Two matrices are **equal** if they have the same dimension, and corresponding elements are the same.

Example 3: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then A and B are equal if and only if $b_{11} = 1$, $b_{12} = 2$, $b_{21} = 3$, and $b_{22} = 4$.

Matrix Addition and Subtraction

Definition: Let A and B be two matrices of the same dimension. We define the sum of two matrices of the same size to be the matrix $A + B$ whose elements are the sums of corresponding elements of A and B.

Example 4:

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1+1 & 2+0 \\ 3+2 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$.
2. Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$, then $C + D = \begin{bmatrix} c_{11}+d_{11} & c_{12}+d_{12} \\ c_{21}+d_{21} & c_{22}+d_{22} \end{bmatrix}$.

Theorem: Properties of matrix addition.

1. Matrix addition satisfies both the associative and commutative properties.

$$A + B = B + A \qquad \textbf{Commutative Property}$$

$$A + (B + C) = (A + B) + C \qquad \textbf{Associative Property}$$

2. There is a **zero matrix** for every dimension. The zero matrix **0** of dimension $m \times n$ is the $m \times n$ matrix whose elements are all zero. For any matrix A,

$$A + 0 = 0 + A = A$$

3. A matrix A can be multiplied by a constant k. The **scalar product** is the matrix kA whose elements are found by multiplying each element of A
4. Each matrix A has an **additive inverse matrix** $-A$ whose elements are the negatives of the elements of A.

Definition: Let A and B be two matrices of the same dimension. Define the **difference of two matrices of the same size** to be the matrix $A-B$ whose elements are the sums of corresponding elements of A and $-B$, that is,

$$A-B=A+(-B)$$

Example 5:

1. Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$, then $-B=\begin{bmatrix}-1 & 0\\ -2 & 1\end{bmatrix}$ and $A-B=\begin{bmatrix}0 & 2\\ 1 & 5\end{bmatrix}$.
2. Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, then $5A=\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$.

Matrix Multiplication

Definition: The product of a row vector and a column vector. The **product** of a $1\times n$ row vector and a $n\times 1$ column vector is a scalar given by

$$\overset{1\times n}{\begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix}}\overset{n\times 1}{\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_4\end{bmatrix}}=\begin{bmatrix}a_1b_1 & +a_2b_2+ & \cdots + & a_nb_n\end{bmatrix}$$

This is the usual dot product of the two vectors

Example 6: Let $A=\begin{bmatrix}2 & 3 & 0 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-1\\ 2\\ 1\\ 0\end{bmatrix}$, then the product AB is the real number $AB=\begin{bmatrix}2(-1)+ & 3(2)+ & 0(1) & -2(0)\end{bmatrix}=[-2+6]=[4]$.

Definition: Let A and B be matrices with dimensions $m \times p$ and $p \times n$, respectively. Define the **product of two matrices** to be the $m \times n$ matrix $C = AB$ whose elements c_{ij} are found by multiplying the ith row of matrix A with the jth column of matrix B.
The number of columns of the first matrix must match the number of rows in the second matrix.

Example 7:

a) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then the product AB is a 2×2 matrix given by

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

b) Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, then the product CD is given by

$$CD = \begin{bmatrix} 1(0)+2(1)+3(1) & 1(1)+2(0)+3(-1) & 1(1)+2(-1)+3(-1) \\ 1(0)+0(1)+-1(1) & 1(1)+0(0)-1(-1) & 1(1)+0(-1)-1(-1) \\ 1(0)+-1(1)-2(1) & 1(1)-1(0)-2(-1) & 1(1)-1(-1)-2(-1) \end{bmatrix} = \begin{bmatrix} 5 & -2 & -4 \\ -1 & 2 & 2 \\ -3 & 3 & 4 \end{bmatrix}$$

Inverse Matrices

Theorem: Properties of Matrix Operations. For matrices A, B, C, 0 (the zero matrix), and I (the identity matrix) of the appropriate dimension (so the product or sum is defined), the following properties hold:

Properties of Matrix Addition

Associative Property: $(A + B) + C = A + (B + C)$

Commutative Property: $A + B = B + A$

Additive Identity Property: $A + 0 = 0 + A = 0$

Additive Inverse Property: $A + (-A) = (-A) + A = 0$

Properties of Matrix Multiplication

Associative Property: $(AB)C = A(BC)$

Multiplicative Identity Property: $AI = IA = A$

Multiplicative Inverse Property: If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$ **.** In this case, A is said to be **invertible.**

Distributive Properties of Matrices

Left Distributive Property: $A(B+C) = AB + AC$

Right Distributive Property: $(B+C)A = BA + CA$

Properties of Equality

Addition Property of Equality: If $A = B$, then $A + C = B + C$.

Left Multiplication Property of Equality:

If $A = B$, then $CA = CB$.

Right Multiplication Property of Equality:

If $A = B$, then $AC = BC$.

Solving a matrix equation

Let A be an $n \times n$ square matrix whose inverse exists. Furthermore, suppose that X and B are both $n \times 1$ column matrices. To solve the matrix equation $AX = B$, use the properties of matrix operations defined earlier:

$$
\begin{aligned}
AX &= B \\
A^{-1}(AX) &= A^{-1}B \\
(A^{-1}A)X &= A^{-1}B \\
IX &= A^{-1}B \\
X &= A^{-1}B
\end{aligned}
$$

To solve the matrix equation, we find A^{-1} and then perform the matrix multiplication $A^{-1}B$. Remember that matrix multiplication is not commutative so the order *must* be correct.

Systems of Equations and Matrix Equations

We can now examine how matrix equations are related to systems of linear equations.

Example 8: The system of linear equations $\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}$ can be written as a matrix equation in the form $AX = B$ like this: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. Then, if A^{-1} exists, the solution is given by $X = A^{-1}B$.

Theorem: If A is a 2×2 square matrix whose inverse exists, then A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Finding the inverse matrix using augmented matrices

Let a matrix A be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The inverse of A is the matrix $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ that satisfies $AA^{-1} = I$, or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrix multiplication yields the systems of equations

$$\begin{aligned} ae + bg &= 1 \qquad & af + bh &= 0 \\ ce + bg &= 0 \qquad & cf + dh &= 1 \end{aligned}.$$

To solve the systems, we could use the augmented matrices

$$\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right]. \qquad \left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right].$$

Since solving both systems involves reducing the same matrix, why not augment the matrix and write it in the form

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] = [A|I]$$

Reducing the left-hand side of this matrix and performing the corresponding operations on the right-hand side produces the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & e & f \\ 0 & 1 & g & h \end{array}\right] = \left[I\middle|A^{-1}\right].$$

Definition: In linear algebra it is proved that A^{-1} exists if and only if the determinant of the square matrix A is nonzero. If $\det(A) = 0$, then the matrix A is called a **singular matrix.** If $\det(A) \neq 0$, then the matrix A is called a **nonsingular matrix.**

Matrix Functions

A vector whose elements are functions: $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

A matrix whose elements are functions: $A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ & & \\ a_{n1}(t) & & a_{nn}(t) \end{pmatrix}$

The derivatives and integrals of these are as expected:

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt}\right) \textbf{ and } \int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt\right).$$