Matrices and Differential Equations - Part 2

Systems of Linear Equations

We can now examine how matrices can be used to solve systems of linear equations.

Example 1: Solving a system using augmented matrices

The system of linear equations
$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$
can be written as an augmented matrix $\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$. Use this matrix to solve the system of equations.

Example 2: Solving a system using augmented matrices

Solve the system of linear equations
$$\begin{aligned} 3x - 4y + 4z &= 7 \\ x - y - 2z &= 2 \\ 2x - 3y + 6z &= 5 \end{aligned}$$
using an augmented matrix.

Linear Independence

Definition: A set of *k* vectors $\vec{x}_1, \ldots, \vec{x}_k$ is **linearly dependent** if there exists a set of (complex) numbers $c_1, \ldots, c_k$, at least one of which is nonzero such that $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{\mathbf{0}}$. If this equation is true only when all the c_k 's are zero, then the set is said to be **linearly independent**.

Example 3: **Determining if a set of vectors is linearly independent**

Determine whether the vectors $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} -4 \\ 1 \\ 11 \end{bmatrix}$ are linearly independent. If they are dependent, show how they can be expressed as a linear combination.

Solution: We must find nonzero values for c_1, c_2 and c_3 so that $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{\mathbf{0}}$ is true. This equation can be written in the matrix form

$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The system can be solved using the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array}\right]$$

As we have seen, the equation $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{\mathbf{0}}$ can be used to determine if a set of vectors is linearly independent by using augmented matrices. Alternatively, we could use determinants. This equation can be written as a matrix equation of the form $\mathbf{AC} = \mathbf{0}$. Then, if the determinant of $\mathbf{A}$ is nonzero, then $\mathbf{A}^{-1}$ exists and the solution is $\mathbf{C} = \mathbf{A}^{-1}\mathbf{0}$ which means that all the c_k's are zero and the vectors are linearly independent. However, if $\det(\mathbf{A}) = 0$ then there are nonzero solutions and the vectors are linearly dependent.

Example 4: **Determining if a set of vectors is linearly independent using determinants**

Use the determinant to show that the set of vectors in example 3 is linearly dependent.

Preliminaries

Definition: A system of linear first-order differential equations is a system of equations of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

This equation is **homogeneous** if $f_k(t) = 0$ for all k, otherwise it is **nonhomogeneous**. Systems of this form can be written as matrix equations of the form $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$. Homogeneous systems can be written in the form $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

Example 5: Writing a System in Matrix Form

Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and write the homogeneous system of differential equations

$$\begin{aligned}
\frac{dx}{dt} &= 2x + 5y \\
\frac{dy}{dt} &= 5x - 7y
\end{aligned}$$ in matrix form.

Example 6: Writing a System in Matrix Form

Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and write nonhomogeneous system of differential equations

$$\frac{dx}{dt} = 2x + 5y + z + t$$
$$\frac{dy}{dt} = 5x + 7y - z + 10t \text{ in matrix form.}$$
$$\frac{dz}{dt} = 2x - 9y - z + 6t$$

Definition: A **solution vector** on an interval I is a vector $\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ whose entries are differentiable functions satisfying the system above on the interval I.

Example 7: Verifying a solution

Show that $\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ and $\mathbf{X}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ are solutions of $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$.

Initial-Value Problem

Let t_0 be a point on an interval I and $\mathbf{X}(t_0)=\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}$ and $\mathbf{X}_0=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ where $b_1,\ldots,b_n$ are given constants. Then the problem

$$\mathbf{X}'=\mathbf{A}(t)\mathbf{X}+\mathbf{F}(t),\ \mathbf{X}(t_0)=\mathbf{X}_0,$$

is an initial-value problem on the interval. If the entries of the matrices $\mathbf{A}(t)$ and $\mathbf{F}(t)$ are continuous functions on a common interval I containing t_0, then there exists a unique solution of the initial-value problem on the interval.

Theorem: Superposition Principle
Let $\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_k$ be a set of solution vectors of the homogeneous system on an interval I. Then the linear combination $\mathbf{X}=c_1\mathbf{X}_1+c_2\mathbf{X}_2+\cdots+c_k\mathbf{X}_k$ is also a solution on the interval.

Theorem: The Wronskian

Let $\mathbf{X}_1=\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$, $\mathbf{X}_2=\begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}$, …, $\mathbf{X}_n=\begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$ be solution vectors of the homogeneous system on an interval I. Then the set of solution vectors is linearly independent iff the Wronskian $W[\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n]=\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}\neq 0$ for every t in the interval I.

Example 8: Linearly independent solutions

Show that $\mathbf{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ and $\mathbf{X}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$ are linearly independent solutions of $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \mathbf{X}$.

Definition: Fundamental Set of Solutions

Any set $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ of n linearly independent solution vectors of the homogeneous system on an interval I is said to be a **fundamental set of solutions on the interval**.

Theorem: There exists a fundamental set of solutions for the homogeneous system on an interval I.

Theorem: Let $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ be a fundamental set of solutions of the homogeneous system on an interval I. Then the **general solution** of the system on the interval is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$.

Example 9: **General Solution of a System**

The vectors $\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ and $\mathbf{X}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ are linearly independent solutions of $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$. Hence, $\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ and $\mathbf{X}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ form a fundamental set of solutions on the interval. The general solution is then $\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$.