Eigenvalues and Eigenvectors

We turn our attention to solving the homogeneous matrix differential equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad (1.1)$$

where $\mathbf{A}$ is an $n \times n$ matrix of constants. Our previous experience with linear differential equations of the form $y' = ay$ suggest that we look for solutions of the form

$$\mathbf{x}(t) = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t} = \mathbf{v} e^{\lambda t}.$$

We substitute into the differential equation (1.1) to obtain

$$\mathbf{A}\mathbf{v} e^{\lambda t} = \lambda \mathbf{v} e^{\lambda t}.$$

Since $e^{\lambda t} \neq 0$, we may divide both sides by $e^{\lambda t}$, leaving

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \text{ or } (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

The trivial solution of this equation is one for which $\mathbf{v} = \mathbf{0}$. A nontrivial solution of this equation is one for which $\mathbf{v} \neq \mathbf{0}$; this occurs if and only if λ and $\mathbf{v}$ satisfy

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= 0 \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} &= 0 \end{aligned} \qquad (1.2)$$

Definition: The first equation of (1.2) is called the **characteristic equation** of the matrix $\mathbf{A}$. The number λ that satisfies the characteristic equation of matrix $\mathbf{A}$ is called an **eigenvalue** of $\mathbf{A}$, and any nonzero column vector that satisfies the second equation of (1.2) is called an **eigenvector** of $\mathbf{A}$ corresponding to λ.

Example 1: Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$.

Solution: First, find $\mathbf{A} - \lambda\mathbf{I}$. Since $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ and $\lambda\mathbf{I} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, we have $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$. The determinant of this matrix gives us the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(1-\lambda) - 6$$

Next, set this equal to zero and solve:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$(2-\lambda)(1-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

So, $\lambda = -1, 4$

To find the eigenvectors we substitute these values one at a time into equation (1.2) and solve the resulting equation.

$\lambda_1 = -1$: With this choice of λ, the matrix $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$.

So that the matrix equation we must solve is $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation can be solved using the augmented matrix $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right]$. This can be reduced to

$\begin{bmatrix} 3 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ so the solution is $3v_1 + 2v_2 = 0$ or $v_1 = -\frac{2}{3}v_2$. We choose $v_2 = 3$ so that we answer involves integers and we have $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$\lambda_2 = 4$: With this choice of λ: $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$. So the system to be solved is $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The associated augmented matrix is $\begin{bmatrix} -2 & 2 & | & 0 \\ 3 & -3 & | & 0 \end{bmatrix}$.

This can be reduced to $\begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ so the solution is $v_1 - v_2 = 0$ or $v_1 = v_2$. We choose $v_2 = 1$ so that the answer involves integers and we have $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Mathematica:

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A= 2
   3 1
 , 
Eigenvalues
 4
Eigenvectors
 3 ,
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Facts:

1. The expression for the characteristic equation of a matrix $\mathbf{A}$ of degree *n* is a polynomial of degree *n*.
2. The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation. According to the fundamental theorem of algebra, then, an $n \times n$ matrix has exactly *n* eigenvalues, counting multiplicities.
3. To each eigenvalue there must correspond at least one eigenvector. If an eigenvalue has multiplicity *m*, then there may be as many as *m* linearly independent eigenvectors corresponding to it.
4. Eigenvectors corresponding to different eigenvalues are independent.

Steps for Solving a Matrix Differential Equation

1. Find the characteristic equation of $\mathbf{A}$, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
2. Find the eigenvalues of $\mathbf{A}$, which are the roots of the characteristic equation.
3. For each eigenvalues λ, find as many linearly independent eigenvectors as possible.
4. The solution of the matrix differential equation takes the form:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

Example 2: **Solving a system of two linear differential equations using eigenvalues**

Solve $\begin{aligned}\frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y\end{aligned}$.

The system of equations has the form $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$. In Example 1, we found the eigenvalues of this matrix to be $\lambda_1 = -1$ and $\lambda_2 = 4$. The corresponding eigenvectors were $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

The solution of the equation is therefore

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} -2c_1 e^{-t} + c_2 e^{4t} \\ 3c_1 e^{-t} + c_2 e^{4t} \end{bmatrix}.$$

Mathematica:

[illegible]

Example 3: **Solving a system of two linear differential equations using eigenvalues**

Solve $\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x + y\end{aligned}$.

Solution: The system of equations has the form $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. The eigenvalues of this matrix are given by the solutions of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Since $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ and $\lambda\mathbf{I} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, we have $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$. Setting the determinant of this matrix equal to zero gives: $\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(1-\lambda) - 4 = 0$.

Multiplying out, we have $\lambda^2 - 2\lambda - 3 = 0$. So, the two eigenvalues are $\lambda = -1, 3$ or $\lambda_1 = 3$ and $\lambda_2 = -1$.

Next, find the eigenvectors.

$\lambda_2 = -1$: With this choice of λ, the matrix $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

So that the matrix equation we must solve is $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation can be solved using the augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]$. This can be reduced to $\left[\begin{array}{cc|c} 1 & .5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so the solution is $v_1 + \frac{1}{2}v_2 = 0$ or $v_1 = -\frac{1}{2}v_2$. We choose $v_2 = -2$ so that we answer involves integers and we have $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\lambda_1 = 3$: With this choice of λ: $\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$. So the system to be solved is $(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The associated augmented matrix is $\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$. This can be reduced to $\left[\begin{array}{cc|c} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ so the solution is $v_1 - \frac{1}{2} v_2 = 0$ or $v_1 = \frac{1}{2} v_2$. We choose $v_2 = 2$ then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The solutions are then $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{y}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$.

We check the Wronskian to see if the solutions form a fundamental set:

$$W = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = e^{-t}\left(2e^{3t}\right) - \left(-2e^{-t}\right)e^{3t} = 4e^{2t} \neq 0 .$$

Therefore the solutions are independent and the general solution of the system is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} .$$

Example 4: **Solving a system of two linear differential equations using eigenvalues**

Solve $\begin{array}{l} \dfrac{dx}{dt} = -2x - 2y \\ \dfrac{dy}{dt} = -x - 3y \end{array}$.

Three Types of Equilibrium Points

If a linear system of two differential equations has two nonzero, distinct, real eigenvalues λ_1, λ_2, then:

- If $\lambda_1 < 0 < \lambda_2$, then the origin is a **saddle**. There are two lines in the phase plane that correspond to straight-line solutions.
- If $\lambda_1 < \lambda_2 < 0$, then the origin is a **sink** (stable node). All solutions tend to $(0,0)$ as $t \to \infty$ and most go to zero in the direction of the λ_2-eigenvectors.
- If $0 < \lambda_2 < \lambda_1$, then the origin is a **source** (unstable node). All solutions except the equilibrium solution tend to infinity as $t \to \infty$ and most solutions leave the origin in the direction of the λ_2-eigenvectors.

Example 5: **Solving a system of three linear differential equations using eigenvalues**

Solve the system of three linear differential equations
$$\begin{aligned}\frac{dx}{dt} &= y + z \\ \frac{dy}{dt} &= x + z \\ \frac{dz}{dt} &= x + y\end{aligned}.$$

Solution: First, write the system as a matrix equation: $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$.

Solutions involving Complex Eigenvalues

Example 6: **Solving a system with complex eigenvalues**

Solve the system $\begin{aligned} \frac{dx}{dt} &= -2x - 3y \\ \frac{dy}{dt} &= 3x - 2y \end{aligned}$

Solution: The characteristic equation of this system is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (-2 - \lambda)(-2 - \lambda) + 9 = 0$$

which simplifies to $\lambda^2 + 4\lambda + 13 = 0$. The eigenvalues are $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$.

To find the eigenvector corresponding to $\lambda_1 = -2 - 3i$ we substitute into the equation $(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = \mathbf{0}$ to get $(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = \begin{bmatrix} -3i & -3 \\ 3 & 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Euler's Formula: $e^{ib} = \cos b + i \sin b$ $\qquad$ $e^{a+ib} = e^{a}(\cos b + i \sin b)$

Continuing with our example, we see that $e^{(-2+3i)t} = e^{-2t}\left[(\cos 3t) + i\sin(3t)\right]$. It follows that the solution has the form:

$$\mathbf{x}(t) = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{-2t}\left[(\cos 3t) + i\sin(3t)\right]$$

$$= \begin{bmatrix} ie^{-2t}\left[(\cos 3t) + i\sin(3t)\right] \\ e^{-2t}\left[(\cos 3t) + i\sin(3t)\right] \end{bmatrix} = \begin{bmatrix} ie^{-2t}\cos(3t) - e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) + ie^{-2t}\sin(3t) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix} + i\begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$$

The two pieces are the real and imaginary parts of the vector function $\mathbf{x}(t)$.

Theorem: If $\mathbf{x}(t)$ is a complex-valued solution to a linear system $\mathbf{x}'(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x}(t)$ where the coefficient matrix has all real entries, and that $\mathbf{x}(t)$ can be written as the sum of real and imaginary parts as $\mathbf{x}(t) = \mathbf{x}(t)_{re} + \mathbf{x}(t)_{im}$ where both $\mathbf{x}(t)_{re}, \mathbf{x}(t)_{im}$ are real-valued functions of *t*. Then $\mathbf{x}(t)_{re}$ and $\mathbf{x}(t)_{im}$ are both solutions of the original system.

Example 6: **Conclusion**

The functions $\mathbf{x}_{re}(t) = \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix}$ and $\mathbf{x}_{im}(t) = \begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$ are both solutions of the system. They are independent since their initial values $\mathbf{x}_{re}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{x}_{im}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are independent. Therefore, the general solution of the system is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{bmatrix}$$