Name Key Test 1A Differential Equations 2009 Mike Huff

Tools: Any calculator but credit will only be awarded for solutions showing work.

(8 points)

1. Solve the initial value problem: $y' = x \ln x$, $y(1) = 0$.

$$\frac{dy}{dx} = x \ln x$$

$$\int dy = \int x \ln x \, dx$$

by parts: $u = \ln x$ $dv = x\,dx$

$du = \frac{1}{x}dx$ $v = \frac{x^2}{2}$

$$y(x) = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$y(x) = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

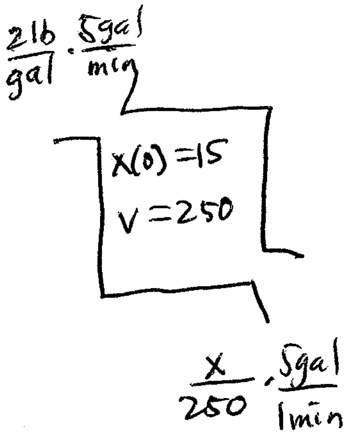
Since $y(1)=0$, we have

$$0 = \frac{1^2}{2}(0) - \frac{1^2}{4} + C \Rightarrow C = \frac{1}{4}$$

$$\therefore \; y(x) = \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4}$$

(8 points)

2. Initially there are 15 pounds of salt dissolved in a tank holding 250 gallons of water. A brine solution containing 2 pounds per gallon is pumped into the tank at a rate of 5 gallons per minute, and a well-stirred mixture is then pumped out at the same rate. Answer the following questions:



a) Find a function $x(t)$ that gives the amount of salt in the tank at any time.

$$\frac{dx}{dt} = 10 - 0.02x$$

$$\int \frac{dx}{10-0.02x} = \int dt$$

$$-50\ln|10-.02x| = t + C$$

$$10 - .02x = Ce^{\frac{1}{-50}t}$$

$$x(t) = 500 - Ce^{\frac{1}{-50}t}$$

$x(0)=15 \Rightarrow C = 485$

$$x(t) = 500 - 485e^{\frac{1}{-50}t}$$

b) Determine the amount of salt in the tank after 27 minutes.

$$x(27) = 500 - 485e^{-\frac{27}{50}} \approx 217.367 \text{ lbs}$$

c) How much salt will be in the tank in the long run, that is, as $t \to \infty$?

$\lim_{t\to\infty} x(t) = 500$ lbs of salt in the tank in the long run.

(8 points)

3. Find the general solution of the differential equation $y \ln y dx - x dy = 0$.

Separable:

$y \ln y \, dx - x \, dy = 0$

$\int \frac{dx}{x} - \int \frac{dy}{y \ln y} = \int 0$

$\ln|x| - \ln|\ln y| = C$

$\int \frac{1}{y \ln y} dy = \int \frac{1}{u} du$

$u = \ln y$

$du \; \frac{1}{y} dy$

$= \ln|u|$

$= \ln|\ln y|$

$\ln\left|\frac{x}{\ln y}\right| = C$

$\frac{x}{\ln y} = C$

$x = C \ln y$

$y = e^{x/c} = e^{cx}$

(8 points)

4. Find the general solution of the differential equation $\frac{dy}{dx} - 2xy = 6xe^{x^2}$.

Linear: $\mu(x) = e^{\int -2x dx} = e^{-x^2}$

$\int (e^{-x^2} y)' = \int 6x \, dx$

$e^{-x^2} y = 3x^2 + C$

$y(x) = 3x^2 e^{x^2} + Ce^{x^2}$

(8 points)

5. Find the solution of the IVP $\frac{dy}{dt} + 2ty = t$. $y(1) = 2$

Linear: $\mu(t) = e^{\int 2t\,dt} = e^{t^2}$

$\int (e^{t^2} y)' = \int t e^{t^2} dt$ $\begin{cases} u = t^2 \\ du = 2t\,dt \\ \frac{1}{2}du = t\,dt \end{cases}$ $\frac{1}{2}\int e^u du$

$e^{t^2} y = \frac{1}{2} e^{t^2} + C$

$y(t) = \frac{1}{2} + Ce^{-t^2}$

$y(1) = 2 \Rightarrow 2 = \frac{1}{2} + Ce^{-1}$

$\frac{3}{2} = Ce^{-1} \Rightarrow C = \frac{3e}{2}$

$\boxed{y(t) = \frac{1}{2} + \frac{3}{2} e^{1-t^2}}$

(8 points)

6. Find a particular solution of the differential equation $y' = (3 - 2y)y$, $y(0) = 1$.

$\frac{dy}{dt} = (3-2y)y$

$\frac{1}{y(3-2y)} = \frac{A}{y} + \frac{B}{3-2y}$ $A = 1/3$, $B = 2/3$

$\int \frac{dy}{y(3-2y)} = \int dt$

$\frac{1}{3}\int \frac{1}{y} dy + \frac{2}{3}\int \frac{1}{3-2y} dy = t + C$

$\frac{1}{3} \ln|y| - \frac{1}{3} \ln|3-2y| = t + C$

$\ln|y| - \ln|3-2y| = 3t + C$

$\ln\left|\frac{y}{3-2y}\right| = 3t + C$

$\frac{y}{3-2y} = Ce^{3t}$

$y = 3ce^{3t} - 2yce^{3t}$

$y + 2yce^{3t} = 3ce^{3t}$

$y(t) = \frac{3ce^{3t}}{1 + 2ce^{3t}}$

$y(t) = \frac{3}{ce^{-3t} + 2} = \frac{3}{2 + ce^{-3t}}$

$y(0) = 1 \Rightarrow c = 1$

$\therefore \boxed{y(t) = \frac{3}{2 + e^{-3t}}}$

(8 points)

7. **Newton's Law of Cooling says that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding environment:**

$$\frac{dT}{dt} = k(T - T_a).$$

We use T to represent the temperature of the object at any time t and T_a for the ambient temperature. A body is heated to $110^\circ C$ and then placed in air at $10^\circ C$. Suppose that the temperature of the body has dropped to $60^\circ C$ after 1 hour. When will it be at $30^\circ C$?

$\frac{dT}{dt} = k(T - T_a)$

$\int \frac{dT}{T-T_a} = \int k\,dt$

$\ln|T - T_a| = kt + C$

$T - T_a = Ce^{kt}$

$T(t) = T_a + Ce^{kt}$

$T(0) = 110^\circ C, \quad T_a = 10^\circ C \quad T(1) = 60^\circ$

$T(t) = 10 + Ce^{kt}$

$T(t) = 10 + 100e^{kt}$

$60 = 10 + 100e^{k} \Rightarrow e^{k} = \frac{1}{2}$

$k = \ln 1/2$

$\boxed{T(t) = 10 + 100e^{t\ln\frac{1}{2}} \approx 10 + 100e^{-.693t}}$

$30^\circ C$ when: $30 = 10 + 100e^{-.693t}$

$\boxed{t = \frac{\ln .2}{\ln(1/2)} \approx 2.32 \text{ hours}}$

(8 points)

8. Find the solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$. Hint: Try substituting $y = vx$

If $y = vx$

then $\frac{dy}{dx} = v(x') + v'x = v + x\frac{dv}{dx}$

So we have

$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{x(1+v)}{x(1-v)}$

$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$

$x\frac{dv}{dx} = \frac{1+v}{1-v} - \frac{v(1-v)}{1(1-v)}$

$x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v}$

$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$

$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$

$\arctan v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$

$\boxed{\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C}$

(8 points)

9. Find the solution [redacted] $(3x^2y^3 + y^4)\,dx + (3x^3y^2 + y^4 + 4xy^3)\,dy = 0$.

Exact?

$M = 3x^2y^3 + y^4$ $\qquad$ $N = 3x^3y^2 + y^4 + 4xy^3$

$M_y = 9x^2y^2 + 4y^3$ $\qquad$ $N_x = 9x^2y^2 + 4y^3$

$\therefore$ It is exact.

$$Q(x,y) = \int M\,dx = \int (3x^2y^3 + y^4)\,dx = x^3y^3 + y^4x + h(y)$$

$$Q_y = N: \quad \cancel{3x^3y^2} + \cancel{4y^3x} + h'(y) = \cancel{3x^3y^2} + y^4 + \cancel{4xy^3}$$

$$h'(y) = y^4$$

$$h(y) = \tfrac{1}{5}y^5$$

$$\therefore\ Q(x,y) = x^3y^3 + xy^4 + \frac{1}{5}y^5 = C$$

(8 points)

10. Write a differential equation that is a mathematical model of the situation described:

a) The time rate of change of population P is inversely proportional to the square of P.

$$\frac{dP}{dt} = \frac{k}{P^2}$$

b) The time rate of change of the velocity v of a skidding car is proportional to the square root of v.

$$\frac{dv}{dt} = k\sqrt{v}$$

c) In a city having a fixed population P of persons, the time rate of change of the number N of those persons who have caught a certain disease is proportional to the number of those who have not caught the disease.

$$\frac{dN}{dt} = k(P-N)$$

d) Solve one of the equations you have written above.

a)

$$\int p^2\,dp = \int k\,dt$$

$$\frac{p^3}{3} = kt + C$$

$$p(t) = \sqrt[3]{3kt + C}$$

b)

$$\int v^{-1/2}\,dv = \int k\,dt$$

$$2\sqrt{v} = kt + C$$

$$\sqrt{v} = \frac{k}{2}t + C$$

$$v(t) = \left(\frac{k}{2}t + C\right)^2$$

c)

$$\int \frac{dN}{P-N} = \int k\,dt$$

$$-\ln|P-N| = kt + C$$

$$P - N = Ce^{-kt}$$

$$N(t) = P - Ce^{-kt}$$

Do one of the following two problems (8 points-each). The other may be done for 5 points extra credit.

11. Suppose a single person carrying the swine flu virus returns to an isolated Texas town of 8,000 people. Assume that the rate at which the virus spreads is directly proportional to both the number x of infected townspeople and to the number of townspeople not infected. Determine the number of infected townspeople at any time t if after 3 days it is further observed that there are 100 students infected: that is $x(3) = 100$. Hint: Use the equation $\frac{dx}{dt} = kx(8000 - x)$.

12. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate $\alpha = 0$) and the death rate (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

$$\frac{dx}{dt} = kx(8000-x)$$

$$\int \frac{dx}{x(8000-x)} = \int k\,dt$$

$$\frac{1}{8000}\left[\int \frac{1}{x}dx + \int \frac{1}{8000-x}dx\right] = kt + C$$

$$\ln|x| - \ln|8000-x| = 8000kt + C$$

$$\left|\frac{x}{8000-x}\right| = Ce^{8000kt}$$

$$x = 8000Ce^{8000kt} - Cxe^{8000kt}$$

$$x = \frac{8000Ce^{8000kt}}{1+Ce^{8000kt}}$$

$$x(t) = \frac{8000}{1+Ce^{-8000kt}}$$

$$\frac{1}{x(8000-x)} = \frac{A}{x} + \frac{B}{8000-x}$$

$$x(0) = 1 \Rightarrow C = 7999$$

$$x(t) = \frac{8000}{1+7999e^{-8000kt}}$$

$$x(3) = 100$$

$$100 = \frac{8000}{1+7999e^{-24000k}}$$

$$100 + 799900e^{-24000k} = 8000$$

$$k = 1.924E-4 = .0001924$$

$$x(t) = \frac{8000}{1+7999e^{-1.5392t}}$$

$(ay-by^2)(b+ay) = aby + a^2y^2 - b^2y^2 - aby^3$

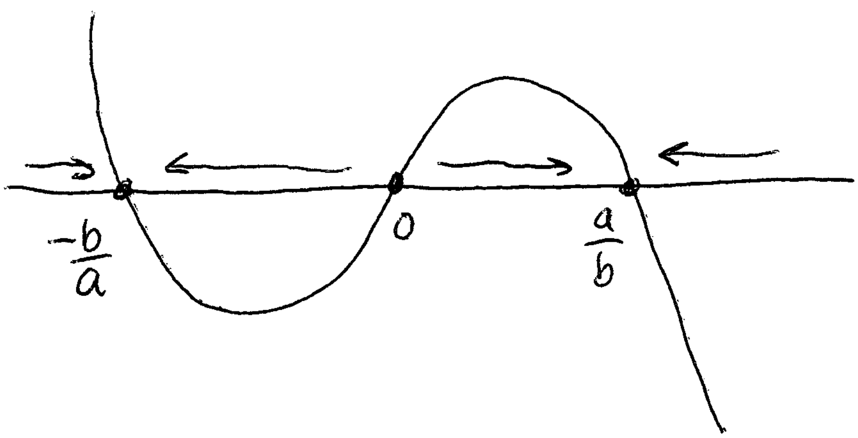
(8 points)

13. For the equation $\frac{dy}{dt} = y\left(a - by\right)(b + ay), 0 < a < b, -\infty < y_0 < \infty$, find the following:

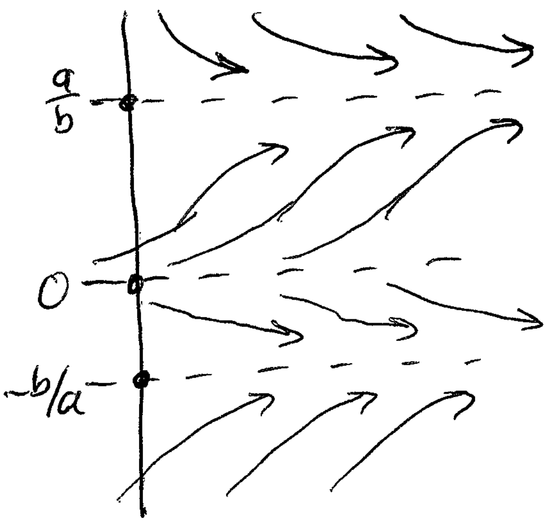
a) The equilibrium solutions
b) Sketch a graph of $f(y)$ versus y.
c) Draw the phase line.
d) Classify the equilibrium solutions as asymptotically stable or unstable

a) $y(a-by)(b+ay) = 0 \Rightarrow \quad y=0, \ y=\frac{a}{b}, \ y=\frac{-b}{a}$

b)



c)



d) $y=0$ unstable
$y=a/b$ stable
$y=-b/a$ stable

(3 points)
14. Sketch the solution curves on the following vector field that correspond to the initial conditions $y(-4) = 1$ and $y(0) = -1$.