Variation of Parameters

We now consider a more general way of solving differential equations of the form

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t)$$

Suppose we have already found the complementary solution of the related homogeneous equation is $ay''+by'+cy=0$, given by:

$$y_c(t)=c_1y_1(t)+c_2y_2(t).$$

In order to find a second solution to the differential equation, this method requires us to replace the constants (parameters) c_1, c_2 with unknown functions $v_1(t), v_2(t)$. With this choice our guess for a particular solution will be of the form

$$y_p=v_1y_1+v_2y_2.$$

Differentiation yields

$$y_p'=\left(v_1'y_1+v_2'y_2\right)+\left(v_1y_1'+v_2y_2'\right).$$

Since $v_1(t)$ and $v_2(t)$ are arbitrary, we may impose two conditions on them. First, we will choose to make y_p a solution of the differential equation; the second condition is chosen to make our work easier. We will choose to let

$$\left(v_1'y_1+v_2'y_2\right)=0. \tag{0.1}$$

Then

$$y_p''=v_1'y_1'+v_2'y_2'+v_1y_1''+v_2y_2''.$$

Substituting into the differential equation, we get

$$a\left(v_1'y_1'+v_2'y_2'+v_1y_1''+v_2y_2''\right)+b\left(v_1y_1'+v_2y_2'\right)+c\left(v_1y_1+v_2y_2\right)=f(t)$$

This can be rearranged to show that

$$v_1\left(ay_1''+by_1'+cy_1\right)+v_2\left(ay_2''+by_2'+cy_2\right)+a\left(v_1'y_1'+v_2'y_2'\right)=f(t). \tag{0.2}$$

Since y_1 and y_2 are solutions of the complementary equation, we know that $ay_1'' + by_1' + cy_1 = 0$ and that $ay_2'' + by_2' + cy_2 = 0$, so equation (0.2) is reduced to

$$a\left(v_1' y_1' + v_2' y_2'\right) = f(t). \tag{0.3}$$

The resulting system of equations (0.1)and (0.3)can now be solved for $v_1'(t)$ and $v_2'(t)$. We may then be able to find $v_1(x)$ and $v_2(x)$ directly using integration.

The system of equations $\left(v_1' y_1 + v_2' y_2\right) = 0$ and $a\left(v_1' y_1' + v_2' y_2'\right) = f(t)$

can be solved using Cramer's rule (here I assume $a = 1$):

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W(y_1, y_2)} \text{ and } v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W(y_1, y_2)}$$

Example 1: Using variation of parameters

Solve $y'' + y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ using variation of parameters

Solution: First solve the related homogeneous equation $y'' + y = 0$. The characteristic equation is $\lambda^2 + 1 = 0$. Which has roots $\lambda = \pm i$. Therefore, the complementary solution is $y_c = c_1 \cos x + c_2 \sin x$. With $y_1 = \cos x$ and $y_2 = \sin x$, we have $y_1' = -\sin x$ and $y_2' = \cos x$. The equations above are then

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \tan x}{1} = -\frac{-\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x \text{, and ,}$$

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ \sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{1} = \sin x$$

Integrating to solve for $v_1(x)$ and $v_2(x)$, we have

$$\int v_1(x)dx = \int (\cos x - \sec x)dx = \sin x - \ln|\sec x + \tan x|$$

$$\int v_2(x)dx = \int \sin x dx = -\cos x\,.$$

Therefore,

$$\begin{aligned} y_p = v_1 y_1 + v_2 y_2 &= (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)\sin x \\ &= -\cos x \ln|\sec x + \tan x|\,, \end{aligned}$$

and the general solution is

$$y(x) = y_c + y_p = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example 2: Using variation of parameters

Solve $y'' + 9y = 9\sec^2 3x$, $0 < x < \dfrac{\pi}{6}$ using variation of parameters

$y_c = c_1 \cos 3x + c_2 \sin 3x$, $y_p = -\sec 3x \cos 3x + \ln|\sec 3x + \tan 3x| \sin 3x$

Example 3: Using variation of parameters

Solve $y'' + 4y' + 4y = x^{-2}e^{-2x}$, $x > 0$ using variation of parameters

$y_p = -e^{-2x}(\ln x - 1)$ or $-e^{-2x}\ln x$ since e^{-2x} appears in y_c