Factoring

Definition: The greatest common factor (GCF) of two or more numbers is the largest number that divides into all of the numbers.

Example: Find the GCF of 24 and 36.

First, we prime factor each number:

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$

Then, select the smallest number in each column and multiply these two numbers together to get the GCF.

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$ Therefore, the GCF of 24 and 36 is $2^2 \cdot 3 = 12$

Exercise: Find the GCF of $52x^2y^2 \text{and} 16xy^3$.

Factoring means writing an expression as a product. In the first part of this section, we will make use of the distributive property to factor out the greatest common factor (GCF) of several terms.

Factoring Out the GCF

Recall that the distributive property is the link between multiplication and addition:

$$ax + ay = a(x + y)$$

↑ addition ↑ multiplication

Example 1: Factor out the GCF

a) $-15x^2y + 9xy$

The GCF is $-3xy$. The expression can be written as

$$-15x^2y + 9xy = -3xy(5x) - 3xy(-3) = -3xy(5x - 3)$$

↑ ↑
Notice that the two terms have a common factor of $-3xy$

b) $14x^3y + 7x^2y - 7xy$

The GCF is $7xy$. The expression can be written as

$$14x^3y + 7x^2y - 7xy = 7xy(2x^2) + 7xy(x) + 7xy(-1) = 7xy(2x^2 + x - 1)$$

c) $4x(3x - 1) + 5(3x - 1)$

In this problem, the GCF is the binomial $(3x - 1)$. When we factor it out, we get:

$$4x(3x - 1) + 5(3x - 1) = (3x - 1)(4x + 5)$$

Exercise 1: Factor out the GCF

a) $8x^2 - 6x^3 + x$

b) $6x^3 - 9x^2 + 12x$

c) $52x^2y^2 - 16xy^3 + 26z$

d) $56xy^5z^{13} - 24y^4z^2$

e) $3(x + y) - 5x(x + y)$

f) $3x^2y(x + 3) - (x + 3)$

Factoring By Grouping

If there are four terms in an algebraic expression, we may be able to factor by grouping.

Example 2: Factoring by Grouping

a) $xy + 2x + 3y + 6$

To factor by grouping, we group the first two and the last two terms together and factor out the GCF from each pair of terms.

$$xy + 2x + 3y + 6 = x(y+2) + 3(y+2)$$

Now we can factor out the common binomial factor $(y+2)$:

$$xy + 2x + 3y + 6 = (x+3)(y+2)$$

b) $x^3 + 4x + x^2 + 4$

$$\underbrace{x^3 + 4x}_{\text{common factor is } x} + \underbrace{x^2 + 4}_{\text{no common factor}} = x\left(x^2 + 4\right) + 1 \cdot \left(x^2 + 4\right) = (x+1)\left(x^2 + 4\right)$$

Exercise 2: Factoring by Grouping

a) $x^3 + 3x^2 + x + 3$

b) $2x^3 - x^2 - 10x + 5$

c) $4x^2 - 8xy - 3x + 6y$

d) $xy - 2yz + 5x - 10z$

e) $x^4 - 5x^3 + 2x^2 - 10x$

Factoring Trinomials of the Form $ax^2 + bx + c$, $a = 1$

Next, we consider how to factor trinomials of the form $x^2 + bx + c$. In order for this to work, we will need two factors such as $(x+n)(x+m)$. If we multiply these together, we get

$$x^2 + bx + c = (x+n)(x+m) = x^2 + nx + mx + nm = x^2 + (n+m)x + nm$$

This shows us that in order to factor a trinomial such as $x^2 + bx + c$, we seek to find two numbers whose sum is *b* and whose product is *c*.

Example 3: Factoring Trinomials

a) Factor $x^2 + 7x + 12$.

Our goal is to write the expression like this: $x^2 + 7x + 12 = (x + \_)(x + \_)$

Since the product of the two numbers we seek is positive, we know that the numbers have the same sign. Since the sum of the two numbers is $+7$, we know that both numbers are positive.

The possible paired factors of 12 are $1 \cdot 12, 2 \cdot 6, 3 \cdot 4$. Since the sum is 7, we know that 3 and 4 are the numbers we seek:

$$x^2 + 7x + 12 = (x+3)(x+4)$$

Check: $(x+3)(x+4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$

b) Factor $x^2 - 8x + 15$.

Since the product of the two numbers we seek is positive, we know that the numbers have the same sign. Since the sum of the two numbers is -8, we know that the numbers are both negative.

$$x^2 - 8x + 15 = (x - \_)(x - \_)$$

The possible factors of 15 are $-1 \cdot (-15), -3 \cdot (-5)$. Since the sum is -8, the correct choice is $-3 \cdot (-5)$:

$$x^2 - 8x + 15 = (x-3)(x-5)$$

Check: $(x-3)(x-5) = x^2 - 5x - 3x + 15 = x^2 - 8x + 15$

c) Factor $x^2 + 2x - 63$.

Since the product of the two numbers we seek is negative, we know that the numbers have opposite sign. Since the sum of the two numbers is $+2$, we know that the larger number is positive.

$$x^2 + 2x - 63 = (x + \_)(x - \_)$$

The possible factors of -63 are $(-1) \cdot 63, (-3) \cdot 21, (-7) \cdot 9$. Since the sum is $+2$, the correct choice is $(-7) \cdot 9$:

$$x^2 + 2x - 63 = (x + 9)(x - 7)$$

Exercise 3: Factoring Trinomials of the Form $x^2 + bx + c$.

a) $x^2 + 4x + 3$

b) $x^2 - 13x + 30$

c) $x^3 - 3x^2 - 28x$ Hint: Remember to always factor out the GCF first.

d) $x^2 + 11xy + 28y^2$

e) $3x^2 - 24x - 60$

f) $5x^3y - 25x^2y^2 - 120xy^3$

Factoring Trinomials of the Form $ax^2 + bx + c$, $a \neq 1$

Next, we consider how to factor trinomials of the form $ax^2 + bx + c$. In order for this to work, we will need two factors such as $(px+n)(qx+m)$. If we multiply these together, we get

$$ax^2 + bx + c = (px+n)(qx+m) = pqx^2 + qnx + pmx + nm = pqx^2 + (qn+pm)x + nm$$

This shows us that in order to factor a trinomial such as $ax^2 + bx + c$, we seek to find four numbers. The product of the first two of these numbers is *c*. The product of the other two is *a*. The sum of the inner and the outer terms is the middle term of the trinomial you are trying to factor.

Example 4: Factoring Trinomials of the Form $ax^2 + bx + c$**.**

a) Factor $2x^2 + 5x + 3$.

Since the 2 and the 3 are both prime, each of these factors in only one way. Since 2 factors into $1 \cdot 2$, we know that the result should look like this:

$$2x^2 + 5x + 3 = (2x + \_)(x + \_)$$

All we have to do now is decide where to put the 3 and the 1. Let's try it both ways:

$2x^2 + 5x + 3 = (2x + 1)(x + 3) = 2x^2 + 7x + 3$ is incorrect.

$2x^2 + 5x + 3 = (2x + 3)(x + 1) = 2x^2 + 5x + 3$ is correct.

b) Factor $3x^2(x-2) + 5x(x-2) - 2(x-2)$.

Since there is a common factor of $(x-2)$, we factor it out first:

$$3x^2(x-2) + 5x(x-2) - 2(x-2) = (x-2)\left(3x^2 + 5x - 2\right)$$

Now the second factor can be factored some more.

$3x^2 + 5x - 2 = (3x + \_)(x - \_)$ or $3x^2 + 5x - 2 = (3x - \_)(x + \_)$

$3x^2 + 5x - 2 = (3x + 2)(x - 1) = 3x^2 - x - 2$ is incorrect

$3x^2 + 5x - 2 = (3x + 1)(x - 2) = 3x^2 - 5x - 2$ is incorrect

$3x^2 + 5x - 2 = (3x - 1)(x + 2) = 3x^2 + 5x - 2$ is correct

Exercise 4: Factoring Trinomials of the Form $ax^2 + bx + c$.

a) Factor $2x^2 + 7x + 5$.

b) Factor $3x^2 + 11x + 6$.

c) Factor $2x^2 + 13x - 7$.

d) Factor $36x^2 + 55x - 14$

Factoring Trinomials of the Form $ax^2 + bx + c$ by Grouping

There is another way of factoring trinomials of this type but it can be very complicated. It uses factoring by grouping.

To factor $ax^2 + bx + c$ by grouping:

Step 1: Find two numbers whose product is *ac* and whose sum is *b*.
Step 2: Write the middle term as the sum of the two numbers found in step 1.
Step 3: Factor the resulting expression using factoring by grouping.

Example 5: Factoring $ax^2 + bx + c$ by Grouping.

a) Factor $3x^2 - 11x - 20$.

Step 1: Find two numbers whose product is $-60 = 3(-20)$ and whose sum is -11.

By trial and error, we discover that -15 and 4 work, since $-60 = 4(-15)$ and $4 + (-15) = -11$

Step 2: Write the middle term as the sum of 4 and -15.

$$3x^2 - 11x - 20 = 3x^2 + 4x - 15x - 20$$

Step 3: Factor the resulting expression using factoring by grouping.

$$3x^2 - 11x - 20 = \underbrace{3x^2 + 4x}_{GCF=x} - \underbrace{15x - 20}_{GCF=-5} = x(3x+4) - 5(3x+4) = (3x+4)(x-5)$$

b) Factor $6x^2 - 19x + 15$.

Step 1: Find two numbers whose product is $90 = 6 \cdot 15$ and whose sum is -19.

By trial and error, we discover that -10 and -9 work, since $90 = -9(-10)$ and $-9 + (-10) = -19$

Step 2: Write the middle term as the sum of -9 and -10.

$$6x^2 - 19x + 15 = 3x^2 - 9x - 10x + 15$$

Step 3: Factor the resulting expression using factoring by grouping.

$$6x^2 - 19x + 15 = \underbrace{6x^2 - 9x}_{GCF=3x} - \underbrace{10x + 15}_{GCF=-5}$$

$$= 3x(2x-3) - 5(2x-3) = (2x-3)(3x-5)$$

Exercise 5: Factoring Trinomials of the Form $ax^2 + bx + c$ by Grouping

a) Factor $8x^2 - 14x + 5$.

b) Factor $3x^2 - x - 10$.

c) Factor $8x^2 + 33x + 4$.

d) Factor $2x^2 + 7x + 6$.

e) Factor $3x^2 + 20x + 12$.

f) Factor $3x^2 + 20x + 12$.

g) Factor $12x^2 + 10xy - 8y^2$.

Special Factoring Formulas

To factor these types of expressions, we will need to recall the special factoring formulas:

Difference of Two Squares: $x^2 - y^2 = (x+y)(x-y)$

Perfect Square Trinomials: $x^2 + 2xy + y^2 = (x+y)^2$

$$x^2 - 2xy + y^2 = (x-y)^2$$

Example 6: Factoring the Difference of Two Squares

a) $x^2 - 49$

First, write the expression in terms of two squares, then use the formula to factor:

$$x^2 - 49 = (x)^2 - 7^2 = (x+7)(x-7)$$

b) $100x^3 - 25x^5$

$100x^3 - 25x^5 = 25x^3(4 - x^2)$ Factor out the GCF first.

$$= 25x^3(2^2 - x^2) = 25x^3(2+x)(2-x)$$

c) $9x^6 - x^4$

$9x^6 - x^4 = x^4(9x^2 - 1) = x^4([3x]^2 - 1^2) = x^4(3x-1)(3x+1)$

d) $100 - (x+5)^2$

$100 - (x+5)^2 = 10^2 - (x+5)^2 = [10 - (x+5)][10 + (x+5)]$

Finally, we can simplify: $100 - (x+5)^2 = (5-x)(15+x)$

Exercise 6: Factoring the Difference of Two Squares

a) $x^2 - 16y^2$

b) $4x^2 - 36y^2$

c) $72y^2 - 16y^2$

d) $(x-y)^2 - 36$

We will now reverse the two special product formulas we learned earlier to make use of these formulas for factoring.

Perfect Square Trinomials: $x^2 + 2xy + y^2 = (x+y)^2$
$x^2 - 2xy + y^2 = (x-y)^2$

Example 7: Factoring Perfect Square Trinomials

a) Factor $x^2 + 14x + 49$.

b) Factor $x^2 + 10xy + 25y^2$.

c) Factor $9x^2 - 30x + 25$.

d) Factor $16x^2 - 80xy + 100y^2$.

Exercise 7: Factoring Perfect Square Trinomials

a) Factor $x^2 + 16x + 64$.

b) Factor $4x^2 + 12x + 9$.

c) Factor $9x^2 - 12x + 4$.

d) Factor $4y^2 - 28xy + 49x^2$.