Solving Quadratic Equations

Quadratic Equations

Definition: An equation is a quadratic equation if it can be written in the form

$$ax^2 + bx + c = 0\,.$$

Exercises:

a) Why can't $a = 0$ in the definition of a quadratic equation?

b) Is the equation $(x-3)(x+2) = 3x$ a quadratic equation?

c) Is the equation $(x-3)(x+3) = x^2$ a quadratic equation?

d) Is the equation $x^3 - 3 = 3x^2$ a quadratic equation?

Factoring

One way to solve quadratic equations is by factoring. This method makes use of the Zero Product Property.

Theorem: Zero Product Property

If a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$ or both a and b are zero.

Examples:

a) $(x-5)(x+2) = 0$

b) $x^2 + 5x + 6 = 0$

c) $x^2 - 5x = 14$

You will need to remember the **special product formulas:**

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
$$(x+y)(x-y) = x^2 - y^2$$

and

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Exercises: Solve the following quadratic equations by factoring.

a) $x^2 - 16 = 0$

b) $2x^2 - x = 3$

c) $x^2 - 3x - 4 = 0$

d) $6x^2 + 7x - 3 = 0$

e) $6x^2 + 11x = 10$

The Square Root Property

Simple quadratic equations of the form $x^2 = k$ can be solved using the square root property.

Theorem: The Square Root Property. If $x^2 = k$ and k is any real number, then $x = \pm\sqrt{k}$.

Notice that, if $k < 0$, the two solutions of the equation are complex conjugate pairs.

Example 1: Solve the quadratic equation $x^2 - 16 = 0$ using the square root property.

First rewrite the equation as $x^2 = 16$. By the square root property, the equation has the solutions $x = \pm\sqrt{16} = \pm 4$. Checking both solutions, we see that they both solve the original equation: $4^2 - 16 = 0$ is true, and

$(-4)^2 - 16 = 0$ is also true.

Exercises: Solve the following quadratic equations using the square root property.

a) $x^2 + 10 = 60$

b) $x^2 - 1 = 0$

c) $(x - 5)^2 = 20$

d) $2(x + 3)^2 + 7 = 17$

e) $3(x - 5)^2 - 7 = 12$

Completing the Square

We will now make use of our ability to solve any equation of the form

$$a(x+h)^2 + k = 0\,. \tag{0.1}$$

A technique known as completing the square will allow us to rewrite any quadratic equation in the form of equation (0.1).

The process of completing the square is related to perfect square trinomials, that is, to trinomials of the form

$$a^2 + 2ab + b^2 = (a+b)^2\,, \text{ or } a^2 - 2ab + b^2 = (a-b)^2\,.$$

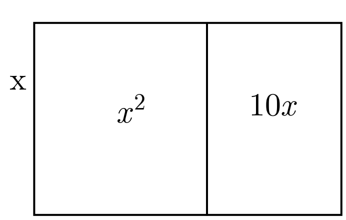
Look for a pattern in the following examples:

- $x^2 + 2x + 1 = (x+1)^2$
- $x^2 + 4x + 4 = (x+2)^2$
- $x^2 + 6x + 9 = (x+3)^2$
- $x^2 + 8x + 16 = (x+4)^2$
- $x^2 + 10x + 25 = (x+5)^2$

How does completing the square work? Let's do an example.

Example 2: Complete the square on the expression $x^2 + 10x$

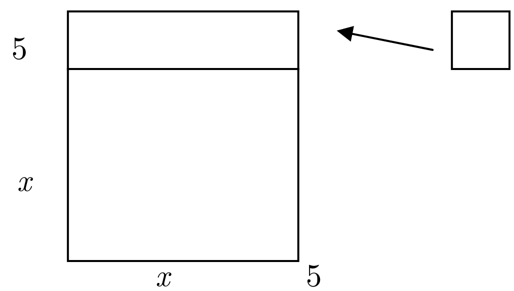
We first consider a geometric interpretation of the situation to show why the process is called completing the square.



The total area of the two rectangles is $x^2 + 10x$.

x 10

If we cut the rectangle on the right in half (take ½ the coefficient of the x term) and move it to the top, the picture becomes:



The area of the little square is 25, the amount we must add to complete the square.

We see that the area becomes $(x+5)^2$ if we add the square of ½ the coefficient of the x term, that is, if we add $\left[\frac{10}{2}\right]^2 = 5^2 = 25$.

How do we use this technique to solve an equation of the form $ax^2 + bx + c = 0$? We can't arbitrarily add a quantity into an equation can we?

Example 3: Solve the equation $x^2 - 6x + 8 = 0$ by completing the square.

1. The first step when completing the square will always be to divide both sides of the equation by the coefficient of the x^2 term to make the leading coefficient one. In this particular problem, that has already been done so we can omit this step.
2. The second step is to use the addition property of equality to move the constant to the right-hand side.

$$x^2 - 6x + \_\_\_ = -8 + \_\_\_$$

3. Take one-half the coefficient of the 1st-degree or x term, square it, and add the result to both sides of the equation using the addition property of equality.

The first-degree term is –6. One-half of –6 is –3. Squaring, we get $(-3)^2 = 9$. Now we can add 9 to both sides of the equation.

$$x^2 - 6x + 9 = -8 + 9$$

4. Factor the left-hand side using the fact that we have constructed a perfect square trinomial.

$$(x-3)^2 = 1$$

5. Use the square root property to solve the resulting equation:

$$\begin{aligned}(x-3)^2 &= 1\\ \sqrt{(x-3)^2} &= \pm\sqrt{1}\\ (x-3) &= \pm 1\\ x &= 3 \pm 1\\ x &= 2, 4\end{aligned}$$

6. Check your answers in the original equation: $x^2 - 6x + 8 = 0$.

Check $x = 2$.

$2^2 - 6 \cdot 2 + 8 = 4 - 12 + 8 = 0$. So $x = 2$ makes the equation true.

Check $x = 4$.

$4^2 - 6 \cdot 4 + 8 = 16 - 24 + 8 = 0$. So $x = 4$ also makes the equation true.

7. The solutions are $x = 2, 4$. So the solution set is $S = \{2, 4\}$.

Exercises: Solve by completing the square.

a) $x^2 - 9x - 14 = 0$
b) $x^2 - 2x - 4 = 0$
c) $-x^2 - 2x + 4 = 0$
d) $-x^2 - 3x + 7 = 0$
e) $2x^2 - 7x + 4 = 0$
f) $4x^2 - 12x + 9 = 0$
g) $\frac{1}{4}x^2 + \frac{3}{4}x - \frac{3}{2} = 0$

One of our main goals in introducing completing the square was to use the technique to solve the general quadratic equation $ax^2 + bx + c = 0$. Solving the general form of this equation by completing the square will produce a formula to solve any quadratic equation in terms of the constants a, b, and c. This result is called the **quadratic formula**

Example 4: The Quadratic Formula.

$ax^2 + bx + c = 0$ Standard form of a quadratic equation.

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ Since $a \neq 0$, we divide by a to make the leading coefficient equal to 1.

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ Use the addition property of equality to move the constant term to the right-hand side.

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ Add the square of one half the coefficient of the first-degree term.

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ By factoring the left-hand side and getting a common denominator on the right-hand side.

$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ Using the square root property.

$\left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ Taking square roots.

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Using the addition property of equality and addition of fractions with a common denominator.

This formula gives us a way of solving all second-degree equations in one variable. Remember that the square root of a negative number is an imaginary number. This means that if the quantity $b^2 - 4ac$ is negative, then the equation $ax^2 + bx + c = 0$ will have two complex conjugate solutions. This important result about the **discriminant,** $b^2 - 4ac$**,** and others are summarized in the table below:

$b^2 - 4ac > 0$	Two real solutions
$b^2 - 4ac = 0$	One real solution (called a double root)
$b^2 - 4ac < 0$	Two complex conjugate pair solutions

Solving Equations Using the Quadratic Formula

The Quadratic Formula: If $ax^2 + bx + c = 0$, and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve an equation using the quadratic formula you must follow these steps:

1) Write the equation in standard form, $ax^2 + bx + c = 0$, and determine the numerical values of a, b, and c.

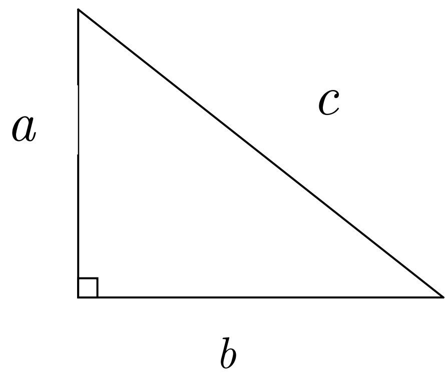
2) Substitute the values for a, b, and c into the quadratic formula and then evaluate the formula to obtain the solution(s).

Exercises: Solve the following equations using the quadratic formula:

a) $x^2 - 6x = -5$
b) $x^2 - 6x = 0$
c) $3x^2 - 4x - 5 = 0$
d) $-6x^2 + 5x + 5 = 0$

Theorem: **The Pythagorean Theorem**. In a right triangle, the sum of the squares of the legs equals the square of the hypotenuse:

$$a^2 + b^2 = c^2$$



Examples: **Applications**:

a) A square television set has a diagonal that is 32". How long is each side of the set?

b) Use the cost equation $C = 0.5x^2 + 15x + 5000$ to find the number of units that can be produced for total costs $C = \$12,000$.

c) The height s, in feet, of a projectile above the ground for any time, t, can be approximated using the formula $s = -16t^2 + v_0 t + s_0$, where v_0 is the initial velocity and s_0 is the projectiles initial height. If a projectile is launched from the ground with an initial velocity of 320 feet per second, its height is described by $s = -16t^2 + 320t$. If the projectile is launched at time $t = 0$, when does it return to the ground?

d) P186 #43

e) P187 #52