Section 21 *Square Roots and Quadratic Equations*

The square root of a number is a number whose square is the given number.

For example, 16 has two square roots, 4 and -4 because $4^2 = 16$ and $\left(-4\right)^2 = 16$.
We use the symbol $\sqrt{x}$ to represent the **positive** number *a* whose square equals *x*. That is, $\sqrt{x} = a$ when $a^2 = x$. The number inside the radical is called the **radicand**.

$\sqrt{16} = 4$ because the symbol $\sqrt{\ }$ means take the positive square root.

Example 1 Finding Square Roots

Find the following:

a. $\sqrt{4}$
b. $\sqrt{9}$
c. $-\sqrt{16}$
d. $\sqrt{-4}$
e. $\sqrt{25}$
f. $\sqrt{225}$

Note: The numbers $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, \ldots$ are called perfect square numbers because each is the square of some positive integer. You should know the square root of each of these numbers. If the radicand is not a perfect square then the radical represents an irrational number.

In general, $\sqrt{x^2} = |x|$. This is true because $\sqrt{2^2} = \sqrt{4} = 2$ but $\sqrt{(-2)^2} = \sqrt{4} = 2$.

Example 2 Evaluating Expressions with Square Roots

Simplify if *x* can represent any real number.

a. $\sqrt{(5x)^2}$
b. $\sqrt{(x-2)^2}$
c. $\sqrt{(2x+1)^2}$

Simplify if *x* represents a nonnegative number.

d. $\sqrt{(2x)^2}$
e. $\sqrt{(2x+3)^2}$

The Square Root Property

Simple quadratic equations of the form $x^2 = k$ can be solved using the square root property.

Theorem: The Square Root Property. If $x^2 = k$ and k is any real number, then $x = \pm\sqrt{k}$.

Notice that, if $k < 0$, the solution is not a real number.

Example 3 Solving quadratic equations using the square root property

Solve $x^2 - 16 = 0$ using the square root property

First rewrite the equation as $x^2 = 16$. By the square root property, the equation has the solutions $x = \pm\sqrt{16} = \pm 4$. Checking both solutions, we see that they both solve the original equation: $4^2 - 16 = 0$ is true, and
$(-4)^2 - 16 = 0$ is also true.

Example 4 Solving quadratic equations using the square root property

Solve the following quadratic equations using the square root property.

a. $x^2 + 10 = 35$

b. $x^2 - 1 = 0$

c. $(x-5)^2 = 7$

d. $2(x+3)^2 + 7 = 17$

Solving Quadratic Equations Using the Quadratic Formula

> The Quadratic Formula: If $ax^2 + bx + c = 0$, and $a \neq 0$, then
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve an equation using the quadratic formula you must follow these steps:

1) Write the equation in standard form, $ax^2 + bx + c = 0$.
2) Determine the numerical values of a, b, and c.
3) Substitute the values for a, b, and c into the quadratic formula and evaluate the formula to obtain the solution(s).

Example 5 Solving quadratic equations using the quadratic formula

Solve the following equations using the quadratic formula:

a. $x^2 - 6x = -5$

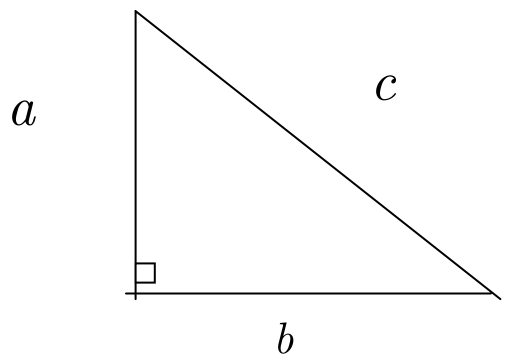
b. $x^2 - 6x = 0$

c. $3x^2 - 4x - 5 = 0$

d. $-6x^2 + 5x + 5 = 0$

Theorem: **The Pythagorean Theorem**. In a right triangle, the sum of the squares of the legs equals the square of the hypotenuse:

$$a^2 + b^2 = c^2$$



Example 6 **Applications involving quadratic equations**

Solve the following applications.

a. A square television set has a diagonal that is 32”. How long is each side of the set?

b. The width of rectangular garden is 5 feet less than the length. If the area of the garden is 84 square feet. Find the dimensions of the garden.

Completing the Square

We will now make use of our ability to solve any equation of the form

$$a(x+h)^2 + k = 0\,. \qquad (0.1)$$

A technique known as completing the square will allow us to rewrite any quadratic equation in the form of equation (0.1).

The process of completing the square is related to perfect square trinomials, that is, to trinomials of the form

$$a^2 + 2ab + b^2 = (a+b)^2, \text{ or } a^2 - 2ab + b^2 = (a-b)^2.$$

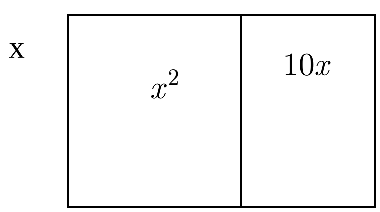
Look for a pattern in the following examples:

- $x^2 + 2x + 1 = (x+1)^2$
- $x^2 + 4x + 4 = (x+2)^2$
- $x^2 + 6x + 9 = (x+3)^2$
- $x^2 + 8x + 16 = (x+4)^2$
- $x^2 + 10x + 25 = (x+5)^2$

How does completing the square work? Let's do an example.

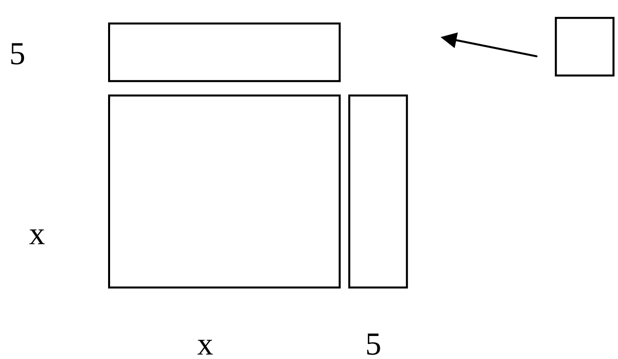
Example 2: Complete the square on the expression $x^2 + 10x$

We first consider a geometric interpretation of the situation to show why the process is called completing the square.



The total area of the two rectangles is $x^2 + 10x$.

If we cut the rectangle on the right in half (take ½ the coefficient of the x term) and move it to the top, the picture becomes:



The area of the little square is 25, the amount we must add to complete the square.

We see that the area becomes $(x+5)^2$ if we add the square of ½ the coefficient of the x term, that is, if we add $\left[\frac{10}{2}\right]^2 = 5^2 = 25$.

How do we use this technique to solve an equation of the form $ax^2 + bx + c = 0$? We can't arbitrarily add a quantity into an equation can we?

Example 3: Solve the equation $x^2 - 6x + 8 = 0$ by completing the square.

1. The first step when completing the square will always be to divide both sides of the equation by the coefficient of the x^2 term to make the leading coefficient one. In this particular problem, that has already been done so we can omit this step.

2. The second step is to use the addition property of equality to move the constant to the right-hand side.

$$x^2 - 6x + \_\_\_ = -8 + \_\_\_$$

3. Take one-half the coefficient of the 1st-degree or x term, square it, and add the result to both sides of the equation using the addition property of equality.

   The first-degree term is –6. One-half of –6 is –3. Squaring, we get $(-3)^2 = 9$.

   Now we can add 9 to both sides of the equation.

$$x^2 - 6x + 9 = -8 + 9$$

4. Factor the left-hand side using the fact that we have constructed a perfect square trinomial.

$$(x-3)^2 = 1$$

5. Use the square root property to solve the resulting equation:

$$(x-3)^2 = 1$$
$$\sqrt{(x-3)^2} = \pm\sqrt{1}$$
$$(x-3) = \pm 1$$
$$x = 3 \pm 1$$
$$x = 2, 4$$

6. Check your answers in the original equation: $x^2 - 6x + 8 = 0$.

   Check $x = 2$.

   $2^2 - 6 \cdot 2 + 8 = 4 - 12 + 8 = 0$. So $x = 2$ makes the equation true.

   Check $x = 4$.

   $4^2 - 6 \cdot 4 + 8 = 16 - 24 + 8 = 0$. So $x = 4$ also makes the equation true.

7. The solutions are $x = 2, 4$.

Exercises: Solve by completing the square.

a) $x^2 - 9x - 14 = 0$
b) $x^2 - 2x - 4 = 0$
c) $-x^2 - 2x + 4 = 0$
d) $-x^2 - 3x + 7 = 0$
e) $2x^2 - 7x + 4 = 0$
f) $4x^2 - 12x + 9 = 0$
g) $\frac{1}{4}x^2 + \frac{3}{4}x - \frac{3}{2} = 0$

One of our main goals in introducing completing the square was to use the technique to solve the general quadratic equation $ax^2 + bx + c = 0$. Solving the general form of this equation by completing the square will produce a formula to solve any quadratic equation in terms of the constants *a*, *b*, and *c*. This result is called the **quadratic formula**

Example 4: The Quadratic Formula.

$ax^2 + bx + c = 0$ Standard form of a quadratic equation

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ Since $a \neq 0$, we divide by a to make the leading coefficient equal to 1

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ Use the addition property of equality to move the constant term to the right-hand side

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ Add the square of one half the coefficient of the first-degree term

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ By factoring the left-hand side and getting a common denominator on the right-hand side

$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ Using the square root property

$\left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ Taking square roots

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Using the addition property of equality and addition of fractions with a common denominator