Section 14 *Systems of Linear Inequalities*

In this section, we examine linear inequalities that contain two variables and systems of inequalities that result from combining two or more inequalities at a time. The solution of an inequality or system of inequalities in two variables is a region in a plane, which consists of a set of ordered pairs (points).

Definition A **linear inequality in two variables** is an inequality that can be written in one of the following standard forms:

$$ax + by < c$$
$$ax + by > c$$
$$ax + by \le c$$
$$ax + by \ge c$$

where *a*, *b*, and *c* are real numbers and *a* and *b* are not both equal to zero.

- A **solution** of a linear inequality in two variables is an ordered pair that when substituted into the linear inequality makes a true statement.
- **Solving** the inequality means finding all the solutions.

Note If one of the variables, *x* or *y*, is raised to a power other than the first power then the resulting inequality is not linear.

Steps in Solving a Linear Inequality in Two Variables

As with other types of inequalities, the process of solving a system of inequalities starts by finding the boundary between regions that include solutions and regions that do not include solutions. Once the boundaries are identified, points within each region defined by the boundary can be tested in the given inequalities to determine whether the region is part of the solution or not. All points within each region defined by the boundary lines will either make the inequality true or make the inequality false, so testing one point within each region will tell you if the region is part of the solution or not. The process is outline below.

Procedure for Solving a Linear Inequality in Two Variables

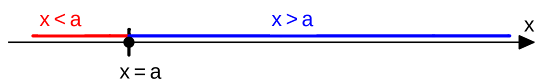
To find the solution of $ax + by < c$, $ax + by > c$, $ax + by \le c$, or $ax + by \ge c$:

- Find the **boundary line**, which is the line $ax + by = c$ that forms the boundary between the ordered pairs that make the inequality true and those that make the inequality false. The boundary line itself may or may not be part of the solution.
  - If the inequality is of the form $ax + by \le c$ or $ax + by \ge c$, then the boundary line $ax + by = c$ is part of the solution and is drawn as a solid line.
  - If the inequality is of the form $ax + by < c$ or $ax + by > c$, then the boundary line $ax + by = c$ is not part of the solution and is drawn as a dotted line.

- To determine which side of the boundary line represents the solution, pick a test point that is not on the boundary line and determine if it makes the inequality true or false. The side which makes the inequality true is shaded. The side of the boundary line that is shaded will represent a **half-plane** of solutions.

Inequalities that Contain One Variable If an inequality contains only the variable *x*, the inequality can be reduced to an inequality of the form $x \le a$ or $x \ge a$ (with the symbols < or > possibly used instead of $\le$ or $\ge$). Such inequalities can be viewed as an interval in one-dimension space or a region in two-dimensional space.

In one-dimension, the value of $x = a$ is the boundary point between the intervals that represent values where $x < a$ and $x > a$, which lie on either side of $x = a$ on a number line. The solutions of $x \le a$ or $x \ge a$ includes the endpoint x = a.



In one dimensional space, the solution of an inequality $x < a$ or $x \le a$ is written in interval notation like (∞, a) or $(\infty, a]$, respectively. Similar statements apply to an inequality $x > a$ or $x \ge a$ is written as (a, ∞) or $[a, \infty)$

In two-dimensional space, the solution is represented by a region on a graph that is expanded vertically to include a second dimension. The boundary between $x < a$ and $x > a$ is the vertical line $x = a$. The solution of $x < a$ includes the region left of $x = a$, and the solution of $x > a$ is the region right of $x = a$. If an inequality includes equality ($\le$ or $\ge$), the values of the coordinates on the line are included in the solution and shown as a solid line. See Figure 1.

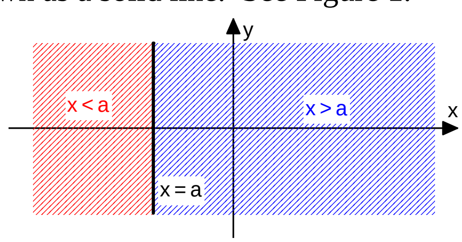


Figure 1 Two-dimensional solutions of $x \le a$ and $x \ge a$

Similarly, inequalities of the form $y \le b$ or $y \ge b$ will have solution regions of a bottom half-plane or a top half-plane shaded, respectively. See Figure 2.

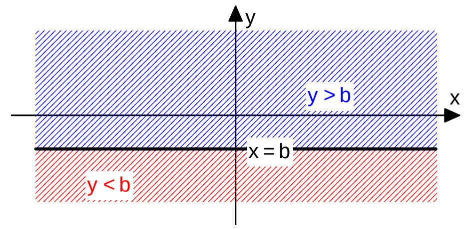


Figure 2 Two-dimensional solutions of $y < b$ and $y > b$

If an inequality statement does not include equality, then the values of the coordinates on the line are not included and the boundary is shown by a dotted line.

Example 1 Graphing a Linear Inequality in Two Variables with One Variable Missing

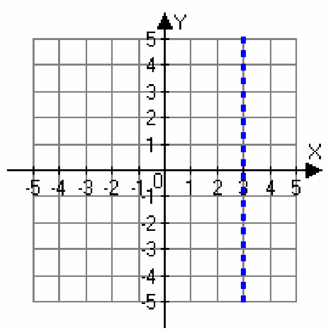
Solve the following linear inequalities in two variables, representing the solution as a shaded region on a graph.

a. $x > 3$ b. $y \le -1$

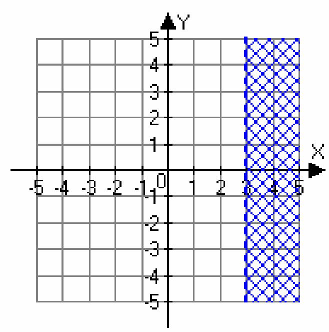
Solution

a. The boundary of the solutions of $x > 3$ is the line $x = 3$, so graph the boundary line $x = 3$. Because the inequality excludes equality, use a dotted line. Next, determine which side of the boundary line represents the solution region. Because $x > 3$, the right side should be shaded.

Sketch the boundary.

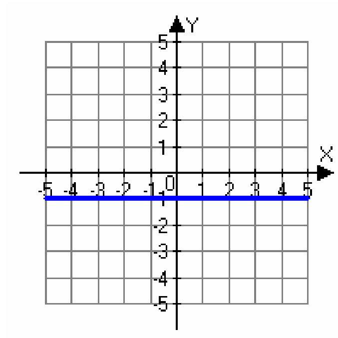


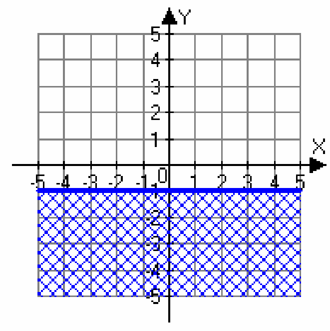
Shade to the right.



b. The boundary of the solutions of $y \le -1$ is the line $y = -1$, so graph the boundary. Because the inequality includes equality, the line should be solid. Next, determine which side of the boundary represents the solution region. Because $y < -1$, the shaded part will be *under* the boundary line.

Sketch the boundary. Shade under the line.





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Try This 1
Solve the following linear inequalities in two variables and represent the solutions as a shaded region on a graph.

a. $x \le -2$ b. $y > 2$

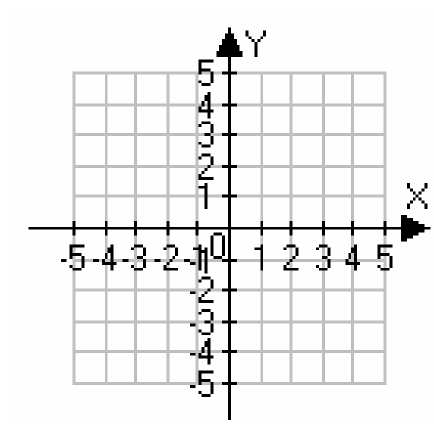
If the inequality contains two variables, care must be taken to ensure that the appropriate side is shaded. This is done by selecting a test point on each side of the boundary line and checking to see if the point makes the inequality true or false, as stated in the following theorem.

Theorem Every point within a region defined by the boundary of an inequality will either be in the solution of the inequality or every point within the region will not be in the solution.

Example 2 Solving a Linear Inequality in Two Variables

Solve the linear inequality in two variables $x + y \le 3$ and represent the solution as a shaded region on a graph.

Solution



Try This 2
Solve the linear inequality in two variables $x - y > -2$ and represent the solution as a shaded region on a graph.

Systems of Linear Inequalities

When two or more linear inequalities are solved at the same time, the result is called a **system of linear inequalities**.

Definition A **system of linear inequalities in two variables** is a collection of inequalities which are to be solved simultaneously.

- A **solution** of a system of linear inequalities in two variables is an ordered pair that makes all the inequalities true.
- **Solving** a system of linear inequalities in two variables means identifying all the ordered pairs that make the inequalities true. Solutions are represented as a shaded region on a graph.

To solve a system of two linear inequalities in two variables, the solution region of each inequality is shaded and then the ordered pairs that make both inequalities true are identified. The ordered pair solutions will be located in the overlapping shaded region(s). More complicated systems containing more inequalities are solved similarly by shading region and locating the ordered pairs that make *all* the inequalities true.

Steps in Solving a System of Linear Inequalities in Two Variables

The steps for solving a system of linear inequalities in two variables are outlined below.

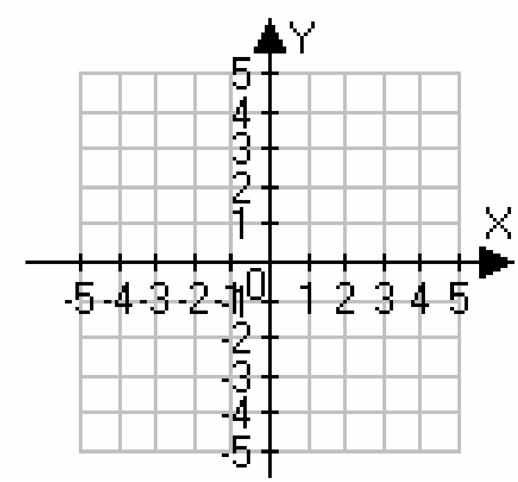
Procedure for Solving a System of Linear Inequalities in Two Variables

- Graph all the boundary lines on the same coordinate plane and shade the appropriate region for each inequality.
- Find and label all the intersection points of the boundary lines.
- Determine the solution region by identifying the overlapped shaded region (if there are any).
- Check by selecting a test point in the solution region and substituting into the original inequalities to make sure they are all true.

Example 3 **Solving a system of two linear inequalities in two variables**

Solve the system of linear inequalities in two variables $\begin{array}{c} -3x + y > -2 \\ 2x + y \le 3 \end{array}$ and represent the solution as a shaded region on a graph.

Solution



Try This 3

Solve the system of linear inequalities in two variables $\begin{aligned} -x+2y &\le 1 \\ -2x+y &> -4 \end{aligned}$ and represent the solution as a shaded region on a graph.

Applications Involving Systems of Linear Inequalities

Some systems with two variables contain more than two inequalities, and they can be solved using the same basic procedure. For example, in many real-world situations the variables must both be nonnegative, as illustrated in Example 4.

Example 4 An Application Involving a System of Linear Inequalities in Two Variables

A company that manufactures two types of DVD players has the following requirements.

1. The basic model requires 2 labor minutes in the manufacturing department and 2.5 labor minutes in the packaging department.
2. The deluxe model requires 5.5 labor minutes in the manufacturing department and 3.5 labor minutes in the packaging department.

The manufacturing department has at most 1000 labor minutes available each day and the packaging department has at most 840 labor minutes available each day.

Find all possible combinations of DVDs that can be made under the current constraints.

Solution

Dictionary
Let x = the number of basic DVD players.
Let y = the number of deluxe DVD players.

Because each basic model requires 2 minutes of labor for manufacturing and each deluxe model requires 5.5 hours of labor for manufacturing and there are a maximum of 480 hours available for labor, the inequality for the manufacturing department is

$$2x+5.5y \le 1000$$ Labor inequality

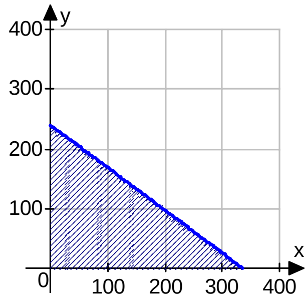
Similarly, the inequality for the packaging department is

$$2.5x+3.5y \le 840$$ Packaging inequality

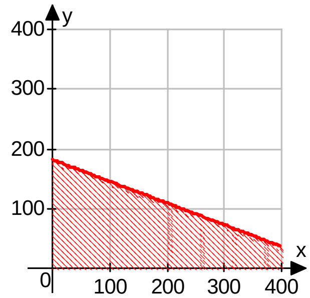
Because the number of DVD players of either type cannot be negative, it must be true that $x \ge 0$ and $y \ge 0$. The system of linear inequalities in two variables to be solved is

$$\begin{aligned} 2x+5.5y &\le 1000 \\ 2.5x+3.5y &\le 840 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

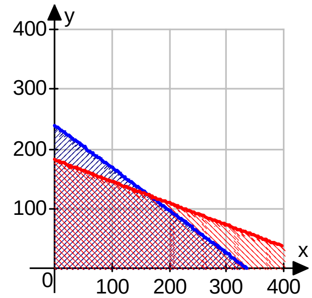
The graph shows the region of the solution of the first inequality
$2x + 5.5y \le 1000$
that lies in the first quadrant where both *x* and *y* are non-negative.



The graph shows the part of the solution of the second inequality
$2.5x + 3.5y \le 840$
that lies in the first quadrant.



The green region of the graph shows the solution region that satisfies all four inequalities.



The cross-hatched shaded region in the graph on the right shows all possible combinations of DVD players that can be made. In reality, however, since a fractional number of players cannot be made, the actual solution includes every ordered pair in the solution region that have whole number coordinates.

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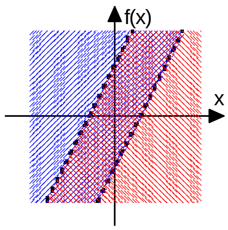
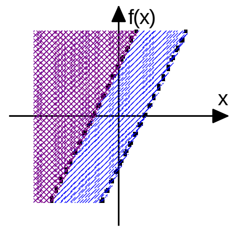
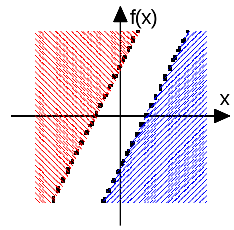
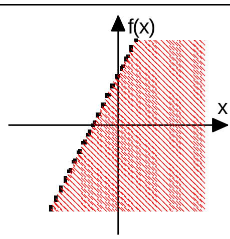
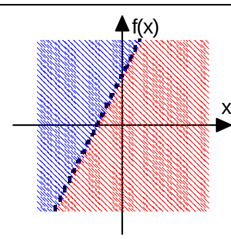
Try This 4

A company makes two types of golf shirts, a plain shirt and a monogrammed shirt. To produce a plain shirt requires 1 hour of sewing time on machine A and 3.5 hours on machine B. To produce a monogrammed shirt, an additional 15 minutes (1/4 of an hours) is needed on machine B. Machine A can be used for at most 50 hours a week and machine B can be used at more 110 hours a week. Find all possible number of the two types of shirts that can be made each week. Represent the solution as a shaded region on a graph.

Special Cases

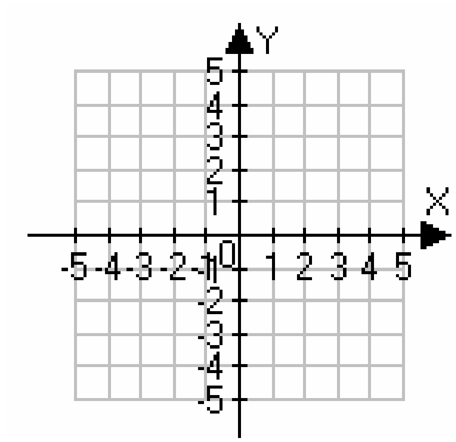
Special cases of systems of inequalities occur when the boundary lines are parallel or the same line (coincident lines). When such situations occur, testing points in each region defined by the boundary lines will indicate which regions are solutions and which are not. Such special cases are summarized in the following table. Solution regions are shown with cross-hatched shading.

Parallel Boundary Lines	 The region between the parallel boundary lines makes both inequalities true.	 The region on one side of one of the boundary lines makes both inequalities true, so the boundary of one inequality defines all of the solutions.	 The individual solution regions occur outside of and on opposite sides of the parallel lines so that no overlapping occurs, therefore there is no solution.
Coincident Boundary Lines	 The solution region occurs on the same side of the boundary lines.	 Individual solution regions occur on opposite sides of the boundary line. No overlapping occurs, so there is no solution.	

Example 5 A system of Linear Inequalities in Two Variables With No Solution

Solve the system of linear inequalities in two variables $\begin{array}{c} 2x+y>2 \\ 6x+3y<2 \end{array}$.

Solution



Try This 5

Solve the system of linear inequalities in two variables $\begin{array}{c} -x+2y\geq -4 \\ x-2y<8 \end{array}$.