Homogeneous

$$(x^2 - xy + y^2)\,dx - xy\,dy = 0$$

$$xy\,dy = x^2 - xy + y^2\,dx$$

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy} \qquad \text{use} \quad y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - x^2v + v^2x^2}{x^2v}$$

$$v + x\frac{dv}{dx} = \frac{1 - v + v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1 - v + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1 - v}{v}$$

$$\frac{v}{1-v}\,dv = \frac{dx}{x} \qquad \frac{v}{1-v} = \frac{v+1-1}{1-v} = \frac{v-1}{-(v-1)} + \frac{1}{v-1}$$

$$\left(\frac{1}{v-1} - 1\right)dv = \frac{dy}{x} \qquad = \frac{1}{v-1} - 1$$

$$\ln|v-1| - v = \ln|x| + C$$

$$\ln\left(\frac{y}{x} - 1\right) - \left(\frac{y}{x}\right) = \ln|x| + C$$

$$\left(\frac{y}{x} - 1\right)e^{-y/x} = Cx$$

Homogeneous Initial Value Problem

$$\frac{dy}{dx} = \frac{2xy + y^2}{x^2 + xy}, \quad y(1) = 1$$

use the substitution $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

so we have

$$v + x\frac{dv}{dx} = \frac{2x^2v + v^2x^2}{x^2 + vx^2} = \frac{2v + v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{2v+v^2}{1+v} - v = \frac{2v + v^2 - v - v^2}{1+v} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v}$$

$$\frac{1+v}{v}\,dv = \frac{dx}{x}$$

$$\left(\frac{1}{v} + 1\right)dv = \frac{dx}{x}$$

$$\ln|v| + v = \ln|x| + C$$

recall that $v = y/x$

$$\ln\left|\frac{y}{x}\right| + \frac{y}{x} = \ln|x| + C$$

Now use the initial condition $y(1) = 1$

$$\ln(1) + 1 = \ln|1| + c \Rightarrow C = 1$$

So the solution is $\ln|y/x| + y/x = \ln|x| + 1$