Separation of Variables

1) Solve the equation

$$(1+y^2)\,dx + (1+x^2)\,dy = 0 \qquad y(0) = -1$$

rewrite as $\dfrac{dy}{dx} = \dfrac{-(1+y^2)}{1+x^2}$

right side of which is cont. and its partial with respect to y is cont near $(0,-1)$ so unique solution exists

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$$

integrate

$$\tan^{-1}x + \tan^{-1}y = C \qquad -\tfrac{1}{2}\pi < \tan^{-1}x < \tfrac{1}{2}\pi$$

$$\tan^{-1}0 = 0 \qquad \tan^{-1}(-1) = -\tfrac{1}{4}\pi$$

$$\Rightarrow \quad C = -\tfrac{1}{4}\pi$$

$$\tan^{-1}(x) + \tan^{-1}(y) = -\tfrac{1}{4}\pi$$

2) Solve the I.V.P

$$2x(y+1)\,dx - y\,dy = 0 \qquad y(0) = -2$$

Separating variables $\quad 2x\,dx = \dfrac{y}{y+1}\,dy$

$$2x\,dx = \left(1 - \frac{1}{y+1}\right)dy$$

$$x^2 = y - \ln|y+1| + C$$

$$0 = -2 - \ln|-1| + C \Rightarrow C = 2$$

$$x^2 = y - \ln|y+1| + 2$$

3) Predator - Prey system

$$\frac{dy}{dx} = \frac{y - xy}{xy - x}$$ seperable

$$\frac{dy}{dx} = \frac{y(1-x)}{x(y-1)} = \left(\frac{y}{y-1}\right)\left(\frac{1-x}{x}\right)$$

$$\int\left(\frac{y-1}{y}\right)dy = \int\left(\frac{1-x}{x}\right)dx$$

$$\int\left(1 - \frac{1}{y}\right)dy = \int\left(\frac{1}{x} - 1\right)dx$$

$$y - \ln|y| = \ln|x| - x + C$$

Seperable

$$\frac{dy}{dx} = \frac{xy + (x - y) - x^2}{xy - y^2}$$

So

$$\frac{dy}{dx} = \frac{-x(x-y) + (x-y)}{y(x-y)} = \frac{1-x}{y}$$

that is

$$\frac{dy}{dx} = \frac{1-x}{y}$$

so that

$$y\,dy = (1-x)\,dx$$